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Radio k-chromatic number of cycles for large k

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For a positive integer k, a *radio* k*-labeling* of a graph G is a function f from its vertex set to the non-negative integers such that for all pairs of distinct vertices u and w, we have $|f(u) - f(w)| \geq k - \mathrm{d}(u, w) + 1$ where $\mathrm{d}(u, w)$ is the distance between the vertices u and w in G. The minimum span over all radio k-labelings of G is called the *radio* k*-chromatic number* and denoted by $rn_k(G)$. The most extensively studied cases are the classic vertex colorings ($k = 1$), $L(2,1)$-labelings ($k = 2$), radio labelings ($k = d$, the diameter of G), and radio antipodal labelings ($k = d - 1$). Determining exact values or tight bounds for $rn_k(G)$ is often non-trivial even within simple families of graphs. We provide general lower bounds for $rn_k(C_n)$ for all cycles C_n when $k \geq d$ and show that these bounds are exact values when $k = d + 1$.

Keywords: Radio k-labeling; radio labeling; radio antipodal labeling; multilevel distance labeling.

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1. Introduction

Given a positive integer k, a function f that assigns a non-negative integer to each vertex of a graph G is called a *radio k-labeling of* G if for any pair of distinct vertices u and w in G, we have

$$|f(u) - f(w)| \geq k - \mathrm{d}(u, w) + 1,$$

where $\mathrm{d}(u, w)$ is the distance between the vertices u and w in G. The *span* of f is the difference between the largest and smallest integers assigned by f. Of particular interest is the *radio k-chromatic number* of G which is the minimum span over all radio k-labelings of G and will be denoted $rn_k(G)$. The radio k-labelings are generalizations of some known graph labelings as shown in Table 1, where d is the diameter of G and each row of the table contains the more standard terminology for the given value of k.

The literature on the radio k-chromatic numbers for $k = 1, 2$ is vast and rich where exact values and tight bounds are known for a large number of families of graphs (for $k = 2$, refer to [7] and the survey [2]). In contrast, not many papers address the cases where $k > 2$, with the majority of them focusing on the cases $k = d - 1, d$. This limited literature may be due to the considerable difficulty in determining $rn_k(G)$ even for graphs as simple as paths and cycles for specific values of $k > 2$. We list some of these results below.

- The radio k-chromatic number of paths on n vertices is known for $k \geq n$, for $k = n - 3$, and for $k = n - 4$ when n is odd and at least 11 [9, 12]; bounds for this number for $k \leq n - 3$ are given in [4].
- A lower bound for the radio k-chromatic number of cycles on n vertices is obtained in [15] for $\lceil (n - 2)/3 \rceil \leq k \leq d$.
- The radio number of paths and cycles are provided in [13].
- The radio antipodal number of paths is found in [11, 12]; the radio antipodal number of cycles is given in [8] except when the number of vertices is a multiple of 4 for which only bounds are presented.
- The radio k-chromatic number of stars is given in [9] and is used to derive an upper bound for the radio k-chromatic number of arbitrary trees.
- A lower bound for the radio number of trees as well as tighter bounds for the radio number of spiders are shown in [5].
- In one of the more recent related papers [16], the radio k-chromatic numbers for $k \geq 2$ of complete multi-partite graphs are determined using an upper bound in

Table 1. Radio k-labelings for $k = 1, 2, d - 1, d$.

k	Radio k-labeling	Radio k-chromatic Number, $rn_k(G)$
1	Classic vertex coloring	Chromatic number, $\chi(G)$
2	L(2,1)-labeling	Lambda number, $\lambda(G)$
$d - 1$	Antipodal labeling	Radio antipodal number, $ac(G)$
d	Radio labeling	Radio number, $rn(G)$

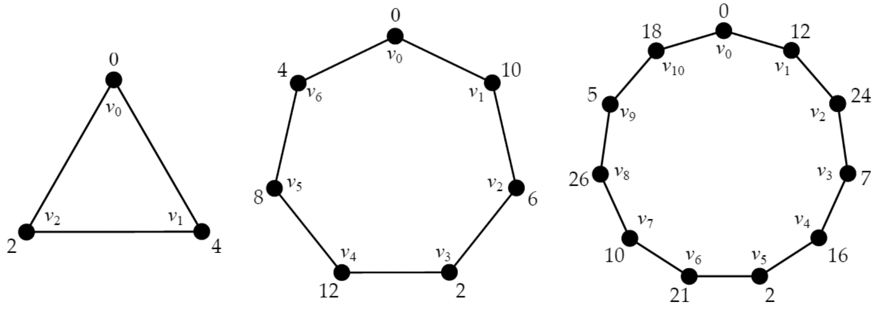

Fig. 1. Near-radio labelings of C_n for $n = 3, 7, 11$ with spans exactly equal to $rn^*(C_n)$.

terms of the path covering number; this result is a generalization of a similar one for the case $k = 2$ in [6].

- Bounds on the radio k-chromatic number for $k \leq d - 2$ are known for powers of cycles [14], for distance graphs [1], for Cartesian products of graphs (select k) [10], and for bipartite graphs [16].
- Bounds on the radio antipodal number of a graph in terms of its order, diameter, and clique number were given in [3].

Inspired by the radio labelings and radio antipodal labelings, we introduce the notion of near-radio labelings, that is, radio k-labelings where k is one greater than the diameter of the graph. More specifically, a *near-radio labeling* of G is a function f from its vertex set to the non-negative integers such that

$$|f(u) - f(w)| \geq d - \mathrm{d}(u, w) + 2$$

for any pair of distinct vertices u and w in G. For simplicity, the $rn_k(G)$ for $k = d+1$ will be denoted $rn^*(G)$. Since $rn^*(P_n)$ where P_n is the path with $n \geq 1$ vertices is known [11, 12], a natural starting point is to focus on $rn^*(C_n)$, where C_n is the cycle with $n \geq 3$ vertices $v_0, v_1, \ldots, v_{n-1}$ such that v_i is adjacent to v_{i+1} for $i = 0, 1, \ldots, n-2$, v_0 is adjacent to v_{n-1}, and the diameter $d = \lfloor n/2 \rfloor$. We were surprised that such a trivial family of graphs provided us with a challenging problem. Figure 1 contains examples of near-radio labelings of C_n for $n = 3, 7, 11$ with spans exactly equal to $rn^*(C_n)$ (exhaustively verified with a computer program).

In this paper, we first find general lower bounds for $rn_k(C_n)$ for all $n \geq 3$ and $k \geq d$ and subsequently use them to determine the exact values for $rn^*(C_n)$ in our main result, Theorem 1.1. The following function was inspired by a similar one introduced by Liu and Zhu [13] in the context of radio labelings and will be used throughout the paper to simplify the exposition of our work (where q is a non-negative integer):

$$\phi(n) = \begin{cases} q+4 & \text{if } n = 4q+2, \\ q+3 & \text{if } n = 4q+r, \text{ where } r = 0, 1, 3. \end{cases}$$

Theorem 1.1. *Let $n = 4q + r \geq 3$ where q and r are integers with $q \geq 0$ and $0 \leq r \leq 3$. Then the following hold*:

(i) $r = 0: \; rn^*(C_n) = \begin{cases} \phi(n)(n-2)/2 + 2 & \text{if } q \text{ is even}, \\ \phi(n)(n-2)/2 + 3 & \text{if } q \text{ is odd}. \end{cases}$

(ii) $r = 1: \; rn^*(C_n) = \phi(n)(n-1)/2.$

(iii) $r = 2: \; rn^*(C_n) = \phi(n)(n-2)/2 + 2.$

(iv) $r = 3: \; rn^*(C_n) = \begin{cases} \phi(n)(n-1)/2 & \text{if } q \neq 2 \text{ is not a multiple of } 3, \\ \phi(n)(n-1)/2 + 1 & \text{otherwise}. \end{cases}$

Throughout this work we will assume $n \geq 3$ and $k \geq d$. In Sec. 2, we provide general lower bounds for $rn_k(C_n)$ which complement the lower bounds provided by Saha and Panigrahi [15] to include the case $k > d$. We begin Sec. 3 by presenting necessary and sufficient conditions for a labeling to be a radio k-labeling of C_n when $k \geq d$. In particular, these conditions simplify similar ones presented by Liu and Zhu [13] in the context of radio labelings. We use this characterization for $k = d+1$ to exhibit near-radio labelings that will achieve the lower bounds for $rn^*(C_n)$ found in Sec. 2, concluding the proof of Theorem 1.1. We offer some closing remarks in Sec. 4.

2. Lower Bounds for $rn_k(C_n)$

In this section, we first derive general lower bounds for $rn_k(C_n)$ by defining a useful function on k and n and by manipulating inequalities due to the definition of radio k-labelings. We then increase these bounds by one for certain combinations of values of k and n. As an application for this general methodology, we use these bounds for $k = d + 1$ to establish lower bounds for $rn^*(C_n)$ which we later show to be exact values in Sec. 3.

Given a radio k-labeling f of C_n, observe that the vertex labels must all be different since we are assuming $k \geq d$. We will use the following conventions through this section:

- $x_0, x_1, \ldots, x_{n-1}$ is the ordering of vertices of C_n where $f(x_i) < f(x_{i+1})$ for $i = 0, 1, \ldots, n-2$; we will assume without loss of generality that $x_0 = v_0$ (otherwise rotate the labels $v_0, v_1, \ldots, v_{n-1}$ around the cycle) and $f(x_0) = 0$;
- π is the permutation so that $x_i = v_{\pi(i)}$ for $i = 0, 1, \ldots, n-1$;
- $f_i = f(x_{i+1}) - f(x_i)$ and $d_i = \mathrm{d}(x_i, x_{i+1})$ for $i = 0, 1, \ldots, n-2$.

Note that $f_i \geq k - d_i + 1$ for $i = 0, 1, \ldots, n-2$ and the span of f is $f(x_{n-1}) - f(x_0) = f_0 + f_1 + \cdots + f_{n-2}$. We illustrate these concepts in Table 2 for the near-radio labeling of C_{11} given in Fig. 1.

Define $\Phi(k, n) = \lceil (3k - n + 3)/2 \rceil$ (observe that $k \geq d$ implies that $3k - n + 3 > 0$). Note that this is a generalization of $\phi(n)$ defined just before Theorem 1.1, in the

Table 2. Near-radio labeling of C_{11} given in Fig.1.

i	0	1	2	3	4	5	6	7	8	9	10
$f(x_i)$	0	2	5	7	10	12	16	18	21	24	26
f_i	2	3	2	3	2	4	2	3	3	2	-
$\pi(i)$	0	5	9	3	7	1	4	10	6	2	8
d_i	5	4	5	4	5	3	5	4	4	5	-

sense that $\Phi(k,n) = \phi(n)$ when $k = d+1$. The first half of Lemma 2.1 presents a relationship between $\Phi(k,n)$ and the sequence $f_0, f_1, \ldots, f_{n-2}$ that will be useful in providing general lower bounds for $rn_k(C_n)$. Liu and Zhu [13] showed a similar result in the context of radio labelings, that is, for $k = d$. Our version extends their result to all $k \geq d$ with a slightly simpler proof. The second half of Lemma 2.1 includes an identity related to the sequence $d_0, d_1, \ldots, d_{n-2}$ that will allow us to improve the lower bounds mentioned earlier for select values of k and n.

Lemma 2.1. *Let f be a radio k-labeling of C_n. For $i = 0, 1, \ldots, n-3$, we have $f_i + f_{i+1} \geq \Phi(k,n)$. In particular, if $f_i + f_{i+1} = \Phi(k,n)$ for an arbitrary i, then $d_i + d_{i+1} = 2k - \Phi(k,n) + 2$ when k and n have different parities.*

Proof. From the definition of radio k-labelings, the following three inequalities hold

$$f_i = f(x_{i+1}) - f(x_i) \geq k - d_i + 1$$

$$f_{i+1} = f(x_{i+2}) - f(x_{i+1}) \geq k - d_{i+1} + 1$$

$$f_i + f_{i+1} = f(x_{i+2}) - f(x_i) \geq k - \mathrm{d}(x_i, x_{i+2}) + 1.$$

Adding these inequalities, we obtain

$$2(f_i + f_{i+1}) \geq 3k - [d_i + d_{i+1} + \mathrm{d}(x_i, x_{i+2})] + 3. \tag{2.1}$$

Consider the path P starting and ending at vertex x_i and following the vertices on the cycle in the direction which ensures that vertex x_{i+1} will precede x_{i+2}. Let ℓ_1, ℓ_2 and ℓ_3 be the lengths of the sections of P from x_i to x_{i+1}, from x_{i+1} to x_{i+2}, and from x_{i+2} to x_i, respectively. Because

$$n = \ell_1 + \ell_2 + \ell_3 \geq d_i + d_{i+1} + \mathrm{d}(x_i, x_{i+2}) \tag{2.2}$$

our earlier inequality (2.1) implies $2(f_i + f_{i+1}) \geq 3k - n + 3$, or $f_i + f_{i+1} \geq \lceil (3k - n + 3)/2 \rceil = \Phi(k,n)$ as desired.

Suppose $f_i + f_{i+1} = \Phi(k,n)$ for an arbitrary $i = 0, 1, \ldots, n-3$. Adding the two inequalities $d_i \geq k - f_i + 1$ and $d_{i+1} \geq k - f_{i+1} + 1$, we obtain $d_i + d_{i+1} \geq 2k - (f_i + f_{i+1}) + 2 = 2k - \Phi(k,n) + 2$. To verify the reverse inequality for the desired values of k and n, note that $f_i + f_{i+1} = \Phi(k,n)$ implies that $f(x_{i+2}) - f(x_i) = \Phi(k,n) \geq k - \mathrm{d}(x_i, x_{i+2}) + 1$ and therefore $\mathrm{d}(x_i, x_{i+2}) \geq k - \Phi(k,n) + 1$. Using the inequality from (2.2), we obtain

$$n \geq d_i + d_{i+1} + \mathrm{d}(x_i, x_{i+2}) \geq d_i + d_{i+1} + k - \Phi(k,n) + 1,$$

which implies $n - k + \Phi(k,n) - 1 \geq d_i + d_{i+1}$. Observe that if k and n have different parities, then $3k - n + 3$ is even, thus $\Phi(k,n) = (3k - n + 3)/2$ which then gives

$$\begin{aligned} 2k - \Phi(k,n) + 2 &= 2k - 2\Phi(k,n) + \Phi(k,n) + 2 \\ &= 2k - (3k - n + 3) + \Phi(k,n) + 2 \\ &= n - k + \Phi(k,n) - 1. \end{aligned}$$

□

Therefore, $d_i + d_{i+1} = 2k - \Phi(k,n) + 2$ as desired.

As a corollary of Lemma 2.1, we find general lower bounds for $rn_k(C_n)$.

Corollary 2.2.

$$rn_k(C_n) \geq \begin{cases} \Phi(k,n)(n-2)/2 + k - d + 1 & \textit{if } n \textit{ even}, \\ \Phi(k,n)(n-1)/2 & \textit{if } n \textit{ odd}. \end{cases}$$

Proof. Let f be a radio k-labeling of C_n with span exactly $rn_k(C_n)$. If n is even, then by Lemma 2.1 we have

$$\begin{aligned} rn_k(C_n) &= (f_0 + f_1) + (f_2 + f_3) + \cdots + (f_{n-4} + f_{n-3}) + f_{n-2} \\ &\geq \Phi(k,n)(n-2)/2 + f_{n-2}. \end{aligned}$$

Therefore, the desired inequality follows since

$$f_{n-2} = f(x_{n-1}) - f(x_{n-2}) \geq k - d_{n-2} + 1 \geq k - d + 1.$$

On the other hand, if n is odd, then again by Lemma 2.1 we have

$$rn_k(C_n) = (f_0 + f_1) + (f_2 + f_3) + \cdots + (f_{n-3} + f_{n-2}) \geq \Phi(k,n)(n-1)/2.$$

□

Observe that if the two inequalities in the previous corollary are tight, it is straightforward to verify that: $f_{2j} + f_{2j+1} = \Phi(k,n)$ for $j = 0, 1, \ldots, \lfloor (n-4)/2 \rfloor$; $f_{n-2} = k - d + 1$ if n is even; and $f_{n-3} + f_{n-2} = \Phi(k,n)$ if n is odd.

The lower bounds given in Corollary 2.2 when $k = d$ are the exact values for the radio number of cycles found by Liu and Zhu [13]. However, for other select values of k and n these lower bounds can be raised by 1 as shown in Propositions 2.5 and 2.6. Before presenting these results, we provide the following auxiliary lemma.

Lemma 2.3. *If n is even and f is a radio k-labeling with span $\Phi(k,n)(n-2)/2 + k - d + 1$, then for $i = 0, 1, \ldots, n-2$ we have*

(i) *$f_i = k - d + 1$ if i even, and $f_i = \Phi(k,n) - (k - d + 1)$ if i odd;*
(ii) *$d_i = d$ if i even, and $d_i = 2k - \Phi(k,n) - d + 2$ if i and k are odd.*

Proof. Let us first verify item (i). From Lemma 2.1, $f_{n-3} + f_{n-2} \geq \Phi(k, n)$. But from the observation made right after Corollary 2.2, $f_{n-2} = k - d + 1$, therefore $f_{n-3} \geq \Phi(k, n) - (k - d + 1)$. In addition, $f_{n-4} + f_{n-3} = \Phi(k, n)$ with $f_{n-4} \geq k - d + 1$, hence $f_{n-3} = \Phi(k, n) - (k - d + 1)$ and $f_{n-4} = k - d + 1$. Replacing n with $n - 2, n - 4, \ldots, 6, 4$ and repeating this process yields the remaining desired values of f_i.

To verify item (ii), let i be an arbitrary even number with $0 \leq i \leq n - 2$. From the definition of radio k-labelings, $d_i \geq k - f_i + 1 = d$ where the last equality follows because $f_i = k - d + 1$ from item (i). Therefore $d_i = d$. If $i \leq n - 3$, $f_i + f_{i+1} = \Phi(k, n)$ from item (i) and since k and n have different parities, Lemma 2.1 implies $d_i + d_{i+1} = 2k - \Phi(k, n) + 2$ and hence $d_{i+1} = 2k - \Phi(k, n) - d_i + 2 = 2k - \Phi(k, n) - d + 2$. □

For the remainder of this work, an arithmetic expression involving integers immediately followed by "(mod n)" indicates that its final value should be taken modulo n, unless the congruence operator "≡" precedes the expression, in which case the standard modular arithmetic conventions apply.

Lemma 2.4. *If k and n have different parities and f is a radio k-labeling of C_n with span exactly equal to the corresponding lower bound presented in Corollary* 2.2, *then* $\pi(i+1) = \pi(i) + d_i \pmod n$ *for all* $i = 0, 1, \ldots, n-2$, *or* $\pi(i+1) = \pi(i) - d_i \pmod n$ *for all* $i = 0, 1, \ldots, n - 2$ (*recall π is the permutation so that $x_i = v_{\pi(i)}$ for* $i = 0, 1, \ldots, n - 1$ *and* $x_0 = v_0$).

Proof. First observe that $d_i = \mathrm{d}(x_i, x_{i+1}) = \mathrm{d}(v_{\pi(i)}, v_{\pi(i+1)})$ which implies $\pi(i + 1) = \pi(i) + d_i \pmod n$ or $\pi(i + 1) = \pi(i) - d_i \pmod n$ for each $i = 0, 1, \ldots, n - 2$.

Suppose n is even and k is odd. Note that if i is even, then Lemma 2.3 implies that $d_i = d$ and so, because $n = 2d$, we have $\pi(i) - d_i \equiv \pi(i) + d_i \pmod n$. If there exists an odd j where $0 < j < n - 4$ so that $\pi(j + 1) = \pi(j) + cd_j \pmod n$ and $\pi(j + 3) = \pi(j + 2) - cd_{j+2} \pmod n$ where $c = \pm 1$, then $d_{j-1} = d_{j+1} = d$ and $d_j = d_{j+2} = 2k - \Phi(k, n) - d + 2$ from Lemma 2.3, hence

$$\begin{aligned}\pi(j + 3) &= \pi(j - 1) + d_{j-1} + cd_j + d_{j+1} - cd_{j+2} \pmod n \\ &= \pi(j - 1) + 2d \pmod n = \pi(j - 1),\end{aligned}$$

which is impossible as π is a permutation. Therefore, such j does not exist and the proposition follows.

Now, suppose n is odd and k is even. We will initially show that for i even and $0 \leq i \leq n - 3$, if $\pi(i + 1) = \pi(i) + d_i \pmod n$, then $\pi(i + 2) = \pi(i + 1) + d_{i+1} \pmod n$. Suppose by contradiction that $\pi(i+2) = \pi(i+1) - d_{i+1} \pmod n$. From the observation made right after Corollary 2.2, we have that $f_i + f_{i+1} = \Phi(k, n)$ so from Lemma 2.1 we obtain $d_i + d_{i+1} = 2k - \Phi(k, n) + 2 = (k + n + 1)/2$. We may assume without loss of generality that $d_{i+1} \geq d_i$ (otherwise switch the roles of d_i and d_{i+1} in the discussion below, excluding the identities involving π). If $d_i \leq k/2$, then $d_{i+1} = (k+n+1)/2 - d_i \geq (n+1)/2 > (n-1)/2 = d$, which is impossible. If on the other hand $d_i > k/2$, then

$$\mathrm{d}(x_i, x_{i+2}) = \mathrm{d}(v_{\pi(i)}, v_{\pi(i+2)}) \leq d_{i+1} - d_i < d_{i+1} - k/2 \leq d - k/2.$$

(The first inequality follows because $\pi(i+2) = \pi(i) + d_i - d_{i+1} \pmod{n}$ and $d_{i+1} \geq d_i$.) But this implies

$$k - \mathrm{d}(x_i, x_{i+2}) + 1 > k - (d - k/2) + 1 = \Phi(k,n) = f_i + f_{i+1} = f(x_{i+2}) - f(x_i),$$

which contradicts the fact that f is a radio k-labeling, so we must have $\pi(i+2) = \pi(i+1) + d_{i+1} \pmod{n}$. Similarly, we can also show that for i even and $0 \leq i \leq n-3$, if $\pi(i+1) = \pi(i) - d_i \pmod{n}$, then $\pi(i+2) = \pi(i+1) - d_{i+1} \pmod{n}$. If there exists an even j where $0 \leq j < n-4$ so that

$$\pi(j+1) = \pi(j) + cd_j \pmod{n},$$
$$\pi(j+3) = \pi(j+2) - cd_{j+2} \pmod{n},$$

where $c = \pm 1$, then

$$\pi(j+2) = \pi(j+1) + cd_{j+1} \pmod{n},$$
$$\pi(j+4) = \pi(j+3) - cd_{j+3} \pmod{n}$$

and therefore

$$\begin{aligned}\pi(j+4) &= \pi(j) + (d_j + d_{j+1}) - (d_{j+2} + d_{j+3}) \pmod{n} \\ &= \pi(j) + (k+n+1)/2 - (k+n+1)/2 \pmod{n} = \pi(j),\end{aligned}$$

which is impossible as π is a permutation. Therefore, such j does not exist, and the proposition follows. □

Proposition 2.5. *If k and n have different parities and* $\gcd(n, 2k - \Phi(k,n) + 2) > 2$, *then*

$$rn_k(C_n) \geq \begin{cases} \Phi(k,n)(n-2)/2 + k - d + 2 & \text{if } n \text{ even}, \\ \Phi(k,n)(n-1)/2 + 1 & \text{if } n \text{ odd}. \end{cases}$$

Proof. We will argue by contradiction that there exists a radio k-labeling f with span exactly equal to the corresponding lower bound in Corollary 2.2. By Lemma 2.1 and the observation following Corollary 2.2, we have $d_{2j} + d_{2j+1} = 2k - \Phi(k,n) + 2$ for $j = 0, 1, \ldots, \lfloor (n-4)/2 \rfloor$, and, if n odd, $d_{n-3} + d_{n-2} = 2k - \Phi(k,n) + 2$. We may assume without loss of generality that $\pi(1) = \pi(0) + d_0 \pmod{n}$ (otherwise reverse the order of vertices on the cycle). From Lemma 2.4, $\pi(i+1) = \pi(i) + d_i \pmod{n}$ for all $i = 0, 1, \ldots, n-1$. Let $\gcd(n, 2k - \Phi(k,n) + 2) = t > 2$ and choose $\ell = n/t - 1$.

Observe that $2 \leq 2\ell + 2 \leq n - 1$ (the second inequality is true since $n \geq 3$ and $t > 2$) and

$$\begin{aligned}\pi(2\ell+2) &= \pi(0) + (d_0 + d_1) + (d_2 + d_3) + \cdots + (d_{2\ell} + d_{2\ell+1}) \pmod{n}\\ &= \pi(0) + (\ell+1)(2k - \Phi(k,n) + 2) \pmod{n}\\ &= \pi(0) + n(2k - \Phi(k,n) + 2)/t \pmod{n} = \pi(0),\end{aligned}$$

which is impossible as π is a permutation. Therefore, the proposition must hold. □

The first lower bound in Proposition 2.5 also holds for some other combinations of odd k and even n without having the gcd requirement satisfied, as shown in Proposition 2.6.

Proposition 2.6. *If $n = 4q$ where q is a positive integer and and $k \equiv 3 \pmod 4$, then*

$$rn_k(C_n) \geq \Phi(k,n)(n-2)/2 + k - d + 2.$$

Proof. Suppose by contradiction that $rn_k(C_n) < \Phi(k,n)(n-2)/2 + k - d + 2$. By Corollary 2.2, there exists a radio k-labeling f with span $\Phi(k,n)(n-2)/2+k-d+1$. Since n is even and k is odd, Lemma 2.3 implies that for $i = 0, 1, \ldots, n-2$: $d_i = d = 2q$ if i is even; and $d_i = 2k - \Phi(k,n) - d + 2 = (k+1)/2$ if i is odd.

We may assume without loss of generality that $\pi(1) = \pi(0) + d_0 \pmod n$ (otherwise, reverse the order of vertices on the cycle). From Lemma 2.4, $\pi(i+1) = \pi(i) + d_i \pmod n$ for all $i = 0, 1, \ldots, n-2$. Therefore, $\pi(i)$ is even for $i = 0, 1, \ldots, n-1$ because n and all d_i are even (note that $k \equiv 3 \pmod 4$ implies that $(k+1)/2$ is even). But this contradicts the fact that π is a permutation of $0, 1, \ldots, n-1$. □

We use Corollary 2.2, Propositions 2.5, and 2.6 to provide the lower bounds of $rn^*(C_n)$ in Corollary 2.7.

Corollary 2.7. *Let $n = 4q+r$ where q and r are integers with $q \geq 0$ and $0 \leq r \leq 3$. Then the following hold*

(i) $r = 0 : rn^*(C_n) \geq \begin{cases} \phi(n)(n-2)/2 + 2 & \text{if } q \text{ is even,}\\ \phi(n)(n-2)/2 + 3 & \text{if } q \text{ is odd.}\end{cases}$

(ii) $r = 1 : rn^*(C_n) \geq \phi(n)(n-1)/2.$

(iii) $r = 2 : rn^*(C_n) \geq \phi(n)(n-2)/2 + 2.$

(iv) $r = 3 : rn^*(C_n) \geq \begin{cases} \phi(n)(n-1)/2 & \text{if } q \text{ is not a multiple of } 3,\\ \phi(n)(n-1)/2 + 1 & \text{otherwise.}\end{cases}$

Proof. In the particular case of near-radio labelings, that is $k = d + 1$, we have $\Phi(k,n) = \phi(n)$ and $rn_k(C_n) = rn^*(C_n)$ as defined in Sec. 1.

In (i) and (iii), n is even, so Corollary 2.2 provides the lower bound $\Phi(k,n)(n-2)/2+k-d+1=\phi(n)(n-2)/2+2$ for $rn^*(C_n)$. We can add 1 to this bound in (i) when q is odd, since in this case $k\equiv 3 \pmod 4$ and Proposition 2.6 confirms this larger bound.

In (ii) and (iv), n is odd, so Corollary 2.2 provides the lower bound $\Phi(k,n)(n-1)/2=\phi(n)(n-1)/2$. We can add 1 to this bound in (iv) when q is a multiple of 3, since in this case $\gcd(n,2k-\Phi(k,n)+2)=\gcd(4q+3,3q+3)\geq 3$ and Proposition 2.5 confirms this larger bound. □

3. Exact Values for $rn^*(C_n)$

In this section, we will completely characterize $rn^*(C_n)$ for all $n\geq 3$ by exhibiting near-radio labelings of C_n with spans that meet the lower bounds of Corollary 2.7, thus concluding the proof of Theorem 1.1. We address the cases where $n=4q+r$ for q a positive integer and $r=0,1,2$ in Propositions 3.2–3.4, respectively. Note that Fig. 1 shows near-radio labelings with span exactly $rn^*(C_n)$ where $n=4q+3$ for $q=0,2$, which were verified exhaustively by a computer program. These instances are not included in the results that follow so they were provided separately. The remaining cases where $n=4q+3$ for integers $q\geq 3$ are more complex and are presented in stages in Propositions 3.6–3.8. The following auxiliary result is instrumental in generating general radio k-labelings of C_n.

Lemma 3.1. *Let $f_0,f_1,\ldots,f_{n-2}$ be a sequence of positive integers and let π be a permutation of $\{0,1,\ldots,n-1\}$ where $\pi(0)=0$. Define $x_i=v_{\pi(i)}$ for $i=0,1,\ldots,n-1$, and consider the function f such that $f(x_0)=0$ and $f(x_{i+1})=f(x_i)+f_i$ for $i=0,1,\ldots,n-2$. Therefore, f is a radio k-labeling of C_n if and only if the two items below are satisfied*

(i) $f_i\geq k-\mathrm{d}(x_i,x_{i+1})+1$;
(ii) $f_i+f_{i+1}\geq k-\mathrm{d}(x_i,x_{i+2})+1$.

Proof. If f is a radio k-labeling of C_n, then (i) and (ii) follow from the definition because $f_i=f(x_{i+1})-f(x_i)$ and $f_i+f_{i+1}=f(x_{i+2})-f(x_i)$.

Suppose on the other hand that (i) and (ii) hold. To prove that f is a radio k-labeling of C_n, it is enough to show that if $0\leq i<j\leq n-1$, then $f(x_j)-f(x_i)=f_i+f_{i+1}+\cdots+f_{j-1}\geq k-\mathrm{d}(x_i,x_j)+1$. If $j=i+1$ or $i+2$, then this last inequality is exactly (i) or (ii), respectively. The two cases below complete the proof.

Case 1: $j=i+3$. For $i,i+1$, and $i+2$, the following three inequalities follow from (i):

$$f_i\geq k-\mathrm{d}(x_i,x_{i+1})+1$$

$$f_{i+1}\geq k-\mathrm{d}(x_{i+1},x_{i+2})+1$$

$$f_{i+2}\geq k-\mathrm{d}(x_{i+2},x_{i+3})+1.$$

Adding these inequalities, we obtain

$$\begin{aligned} f_i + f_{i+1} + f_{i+2} &\geq 3k - [\mathrm{d}(x_i, x_{i+1}) + \mathrm{d}(x_{i+1}, x_{i+2}) + \mathrm{d}(x_{i+2}, x_{i+3})] + 3 \\ &\geq 3k - [n - \mathrm{d}(x_i, x_{i+2}) + \mathrm{d}(x_{i+2}, x_{i+3})] + 3 \quad \text{(a)} \\ &\geq 3k - [n + \mathrm{d}(x_i, x_{i+3})] + 3 \quad \text{(b)} \\ &\geq 3k - 2d - \mathrm{d}(x_i, x_{i+3}) + 2 \quad \text{(c)} \\ &\geq k - \mathrm{d}(x_i, x_{i+3}) + 1. \quad \text{(d)} \end{aligned}$$

For each of the respective lower bounds in steps (a) through (d), we used the following facts:

(a) from the proof of Lemma 2.1, we have $n \geq \mathrm{d}(x_i, x_{i+1}) + \mathrm{d}(x_{i+1}, x_{i+2}) + \mathrm{d}(x_i, x_{i+2})$, or equivalently, $n - \mathrm{d}(x_i, x_{i+2}) \geq \mathrm{d}(x_i, x_{i+1}) + \mathrm{d}(x_{i+1}, x_{i+2})$;
(b) from the triangle inequality, we have $\mathrm{d}(x_{i+2}, x_{i+3}) \leq \mathrm{d}(x_i, x_{i+2}) + \mathrm{d}(x_i, x_{i+3})$, or equivalently, $\mathrm{d}(x_i, x_{i+3}) \geq -\mathrm{d}(x_i, x_{i+2}) + \mathrm{d}(x_{i+2}, x_{i+3})$;
(c) $2d + 1 \geq n$;
(d) $k \geq d$.

Case 2: $j \geq i + 4$. As (i) and (ii) hold, the same arguments used in the proof of Lemma 2.1 can be applied here to show that $f_i + f_{i+1} \geq \Phi(k, n)$ and $f_{i+2} + f_{i+3} \geq \Phi(k, n)$, and hence

$$\begin{aligned} f_i + f_{i+1} + \cdots + f_{j-1} &\geq f_i + f_{i+1} + f_{i+2} + f_{i+3} \\ &\geq 2\Phi(k, n) \\ &= 2\lceil (3k - n + 3)/2 \rceil \\ &\geq 3k - n + 3 \\ &\geq k - \mathrm{d}(x_i, x_j) + 1. \end{aligned}$$

Note that the last inequality can be verified as in Case 1 because of the facts given in (c) and (d), and because $\mathrm{d}(x_i, x_j) \geq 1$. □

Note that for the case $k = d$, two additional conditions, other than (i) and (ii) in Lemma 3.1, were mentioned in Liu and Zhu [13], namely: $f_i + f_{i+1} + f_{i+2} \geq d - \mathrm{d}(x_i, x_{i+3}) + 1$ and $f_i + f_{i+1} + f_{i+2} + f_{i+3} \geq d$. However, these are not necessary to conclude that f is a radio labeling of C_n as verified in Lemma 3.1.

To prove each of the propositions mentioned in the first paragraph of this section, we will first exhibit two sequences of positive integers $d_0, d_1, \ldots, d_{n-2}$ and $f_0, f_1, \ldots, f_{n-2}$. Based on these sequences, the *associated functions* π and f are defined as follows (these conventions will be used from this point forward):

- $\pi(0) = 0$ and $\pi(i + 1) = \pi(i) + d_i \pmod{n}$ for $i = 0, 1, \ldots, n - 2$;
- $x_i = v_{\pi(i)}$ for $i = 0, 1, \ldots, n - 1$;
- $f(x_0) = 0$ and $f(x_{i+1}) = f(x_i) + f_i$ for $i = 0, 1, \ldots, n - 2$.

The proof proceeds with the verification that the associated function π is a permutation of $\{0, 1, \ldots, n - 1\}$ so the vertices $x_0, x_1, \ldots, x_{n-1}$ are exactly the vertices of C_n. To finish the proof, we verify that the associated function f satisfies items (i) and (ii) of Lemma 3.1 when $k = d+1$, which implies that f is a near-radio labeling of C_n with span $f(x_{n-1})$. This span turns out to match the lower bound of $rn^*(C_n)$ in the respective item of Corollary 2.7 and therefore it is exact.

Proposition 3.2. *If* $n = 4q$ *where* q *is a positive integer, then*

$$rn^*(C_n) = \begin{cases} \phi(n)(n-2)/2 + 2 & \text{if } q \text{ is even,} \\ \phi(n)(n-2)/2 + 3 & \text{if } q \text{ is odd.} \end{cases}$$

Proof. For $i = 0, 1, \ldots, n-2$, let

$$d_i = \begin{cases} 2q & \text{if } i \text{ even,} \\ q & \text{if } i = 2q-1 \text{ and } q \text{ odd,} \\ q+1 & \text{otherwise.} \end{cases}$$

Observe that the associated function π is equivalent to

$$\pi(2j) = j(3q+1) \pmod n$$

$$\pi(2j+1) = j(3q+1) + 2q \pmod n$$

for $j = 0, 1, \ldots, q-1$, and

$$\pi(2j) = j(3q+1) - (q \bmod 2) \pmod n$$

$$\pi(2j+1) = j(3q+1) + 2q - (q \bmod 2) \pmod n,$$

for $j = q, q+1, \ldots, 2q-1$. Note that when q is odd, $\pi(i)$ is even for $i = 0, 1, \ldots, 2q-1$, and $\pi(i)$ is odd for $i = 2q, 2q+1, \ldots, n-1$.

We first show that π is a permutation of $\{0, 1, \ldots, n-1\}$. Suppose for contradiction that this is not the case. Let j and j' be non-negative integers smaller than $2q$. Because $(3q+1) \equiv -(q-1) \pmod n$, we have $(j-j')(3q+1) \equiv (j'-j)(q-1) \pmod n$. Without loss of generality, let $j' > j$. We have to examine two cases:

Case 1: Suppose $\pi(2j) = \pi(2j')$ or $\pi(2j+1) = \pi(2j'+1)$. From the note on the parities of values of $\pi(i)$, either $0 \le j < j' < q$ or $q \le j < j' < 2q$. Then $(j'-j)(q-1) \equiv 0 \pmod n$. If q is even, then $\gcd(n, q-1) = \gcd(q, q-1) = 1$, so $(j'-j) \equiv 0 \pmod n$. But $0 < j'-j \le 2q-1 < n$, so this is impossible. If q is odd, then $\gcd(n, q-1) = 2$ or 4. Then $\gcd(n/2^t, (q-1)/2^t) = 1$ for some $t = 1, 2$. Then we have $(j'-j)(q-1)/2^t \equiv 0 \pmod{n/2^t}$, so $(j'-j) \equiv 0 \pmod{n/2^t}$. But recall that when q is odd, $\pi(i)$ is even for $i = 0, 1, \ldots, 2q-1$ and $\pi(i)$ is odd for $i = 2q, 2q+1, \ldots, n-1$, so $0 \le j < j' \le q-1$ or $q \le j < j' \le 2q-1$. Hence $0 < j'-j \le q-1 < n/4 \le n/2^t$ which contradicts $(j'-j) \equiv 0 \pmod{n/2^t}$.

Case 2: Suppose $\pi(2j) = \pi(2j'+1)$. Then $(j-j')(3q+1) + 2q \equiv 0 \pmod n$, or equivalently $(j'-j)(q-1) + 2q \equiv 0 \pmod n$. We can rewrite this as $(j'-j)(q-1) + 2q = 4qx$ for some integer x. Then $(j'-j)(q-1) = 2q(2qx-1)$, which yields $(j'-j)(q-1) \equiv 0 \pmod{2q}$. If q is even, then $\gcd(2q, q-1) = \gcd(q, q-1) = 1$, so $(j' - j) \equiv 0 \pmod{2q}$ which is impossible because $0 < j' - j \leq 2q - 1 < 2q$. If q is odd , then $(j' - j)(q - 1)/2 \equiv 0 \pmod{q}$. But $\gcd(q, (q - 1)/2) = 1$, so $(j' - j) \equiv 0 \pmod{q}$ which is impossible because $0 < j' - j < q$ (as shown at the end of Case 1).

Because we reached a contradiction in both cases, we finally conclude that π is a permutation of $\{0, 1, \ldots, n - 1\}$. For $i = 0, 1, \ldots, n - 2$, let $f_i = d - d_i + 2$, or more specifically

$$f_i = \begin{cases} 2 & \text{if } i \text{ even,} \\ q + 2 & \text{if } i = 2q - 1 \text{ and } q \text{ odd,} \\ q + 1 & \text{otherwise.} \end{cases}$$

We have for all i that $\mathrm{d}(x_i, x_{i+1}) = \mathrm{d}(v_{\pi(i)}, v_{\pi(i+1)}) = \min\{d_i, n - d_i\} = d_i$, therefore $f_i = d - \mathrm{d}(x_i, x_{i+1}) + 2$ so item (i) in Lemma 3.1 is satisfied. By inspection, $d_i + d_{i+1} = 3q$ or $3q + 1$, and $f_i + f_{i+1} = q + 3$ or $q + 4$. Since, $d < d_i + d_{i+1} \leq n$, we must have $\mathrm{d}(x_i, x_{i+2}) = n - (d_i + d_{i+1})$. Then,

$$\begin{aligned} f_i + f_{i+1} \geq q + 3 &= (3q + 1) - (2q - 2) \geq (d_i + d_{i+1}) - 2q + 2 \\ &= 2q - [4q - (d_i + d_{i+1})] + 2 = d - [n - (d_i + d_{i+1})] + 2 \\ &= d - \mathrm{d}(x_i, x_{i+2}) + 2. \end{aligned}$$

Thus, item (ii) in Lemma 3.1 is also satisfied, and we can conclude that the associated function f is a near-radio labeling of C_n. The span of f is $f(x_{n-1}) = \phi(n)(n - 2)/2 + 2$ if q is even, and $f(x_{n-1}) = \phi(n)(n - 2)/2 + 3$ if q is odd, so the desired result follows from item (i) of Corollary 2.7. □

The proofs of Propositions 3.3 and 3.4 use the same sequence of positive integers $d_0, d_1, \ldots, d_{n-2}$ and permutation π used by Liu and Zhu [13] when computing the radio numbers of C_n for $n = 4q + 1$ and $n = 4q + 2$, respectively. Therefore, we refer the reader to their work for details on the verifications that π is indeed a permutation of $\{0, 1, \ldots, n - 1\}$.

Proposition 3.3. *If* $n = 4q + 1$ *where* q *is a positive integer, then*

$$rn^*(C_n) = \phi(n)(n - 1)/2.$$

Proof. For $j = 0, 1, \ldots, q - 1$, let $d_{4j} = d_{4j+2} = 2q - j$ and $d_{4j+1} = d_{4j+3} = q + 1 + j$. From Liu and Zhu [13], the associated function π is a permutation of $\{0, 1, \ldots, n - 1\}$.

For $i = 0, 1, \ldots, n - 2$, let $f_i = d - d_i + 2$. It is straightforward to check that items (i) and (ii) from Lemma 3.1 hold (details are left to the reader) so the associated function f is a near-radio labeling of C_n. The span of f is $f(x_{n-1}) = \phi(n)(n - 1)/2$ and the desired result holds from item (ii) of Corollary 2.7. □

Proposition 3.4. *If $n = 4q + 2$ where q is a positive integer, then*

$$rn^*(C_n) = \phi(n)(n-2)/2 + 2.$$

Proof. For $i = 0, 1, \ldots, n-2$, let $d_i = 2q+1$ if i even and $d_i = q+1$ if i odd. From Liu and Zhu [13], the associated function π is a permutation of $\{0, 1, \ldots, n-1\}$.

For $i = 0, 1, \ldots, n-2$, let $f_i = d - d_i + 2$, or more specifically $f_i = 2$ if i even, and $f_i = q + 2$ if i odd. It is straightforward to check that items (i) and (ii) from Lemma 3.1 hold (details are left to the reader) so the associated function f is a near-radio labeling of C_n. The span of f is $f(x_{n-1}) = \phi(n)(n-2)/2 + 2$ and the desired result follows from item (iii) of Corollary 2.7. Therefore, the lemma holds. □

Before we proceed to the more complex case $n = 4q + 3$ for $q \geq 3$, we need an auxiliary result.

Lemma 3.5. *Let a and b be two positive integers such that $\gcd(a, b) = 1$. The function g defined as $g(0) = 0$ and $g(i+1) = g(i) + a \pmod b$ for $i = 0, 1, \ldots, b-2$ is a permutation of $\{0, 1, \ldots, b-1\}$.*

Proof. Let us assume to the contrary that g is not a permutation of $\{0, 1, \ldots, b-1\}$. Hence there are integers j and j' such that $0 \leq j < j' \leq b-1$ so that $g(j) = g(j')$, or equivalently, $(j' - j)a \equiv 0 \pmod b$. Since $\gcd(a, b) = 1$, we must have $(j' - j) \equiv 0 \pmod b$ which is impossible as $0 < j' - j < b$. Therefore, the desired result holds. □

Proposition 3.6. *If $n = 4q + 3$ where q is odd, $q \geq 1$, and q is not a multiple of 3, then*

$$rn^*(C_n) = \phi(n)(n-1)/2.$$

Proof. For $i = 0, 1, \ldots, n-2$, let $d_i = (3q+3)/2$. Note that $\gcd((3q+3)/2, n)$ divides $8(3q+3)/2 - 3n = 3$; since $\gcd((3q+3)/2, n) = 3$ could only hold when q is multiple of 3, we must have $\gcd((3q+3)/2, n) = 1$. Therefore, Lemma 3.5 (with $a = (3q+3)/2$ and $b = n$) shows that π is a permutation of $\{0, 1, \ldots, n-1\}$.

For $i = 0, 1, \ldots, n-2$, let $f_i = d - d_i + 2$ or more specifically $f_i = (q+3)/2$. Observe that

$$\begin{aligned} \mathrm{d}(x_i, x_{i+1}) &= \mathrm{d}(v_{\pi(i)}, v_{\pi(i+1)}) = \min\{d_i, n - d_i\} \\ &= \min\{(3q+3)/2, (5q+3)/2\} = d_i, \end{aligned}$$

and

$$\begin{aligned} \mathrm{d}(x_i, x_{i+2}) &= \mathrm{d}(v_{\pi(i)}, v_{\pi(i+2)}) = \min\{2d_i, n - 2d_i\} \\ &= \min\{3q+3, q\} = q. \end{aligned}$$

Therefore, $f_i = (q+3)/2 = d - \mathrm{d}(x_i, x_{i+1}) + 2$ and $f_i + f_{i+1} = q + 3 = d - \mathrm{d}(x_i, x_{i+2}) + 2$ so items (i) and (ii) in Lemma 3.1 are satisfied, and we conclude that the associated function f is a near-radio labeling of C_n. The span of f is $f(x_{n-1}) = \phi(n)(n-1)/2$ so the desired result follows from item (iv) of Corollary 2.7. □

Proposition 3.7. *If $n = 4q+3$ where q is even, $q \geq 4$, and q is not a multiple of 3, then*

$$rn^*(C_n) = \phi(n)(n-1)/2.$$

Proof. Define $\pi^*(0) = 0$ and $\pi^*(i+1) = \pi^*(i)+3q+3 \pmod n$, for $i = 0, 1, \ldots, n-2$. Since $\gcd(3q+3, n) = 1$, we have from Lemma 3.5 (with $a = 3q+3$ and $b = n$) that π^* is a permutation of $\{0, 1, \ldots, n-1\}$. We will construct another permutation π of $\{0, 1, \ldots, n-1\}$ based on π^* as follows (we are not abusing the notation here as we will later provide the sequence $d_0, d_1, \ldots, d_{n-2}$ so that π is exactly the associated function). Set $s = (n-5)/2$ and define

$$\begin{aligned}
\pi(2i) &= \pi^*(i) \text{ for } i = 0, 1, \ldots, s \\
\pi(2i+1) &= \pi^*(s+5+i) \text{ for } i = 0, 1, \ldots, s-1 \\
\pi(n-4) &= \pi^*(s+3) \\
\pi(n-3) &= \pi^*(s+1) \\
\pi(n-2) &= \pi^*(s+4) \\
\pi(n-1) &= \pi^*(s+2).
\end{aligned}$$

Informally, the permutation π starts by sequentially alternating the first $s+1$ terms of π^* with the last s terms, in order, starting with $\pi^*(0) = 0$; π ends by conveniently arranging the remaining terms $\pi^*(s+j)$ for $j = 1, 2, 3, 4$, to satisfy the requirements of this proof.

For $j = 0, 1, \ldots, (n-7)/2$, let $d_{2j} = 3q/2+3$ and $d_{2j+1} = 3q/2$. In addition, let $d_{n-2} = d_{n-4} = 2q$ and $d_{n-3} = d_{n-5} = q+3$. For $i = 0, 1, \ldots, s-1$, the straightforward computations below, where operations are taken modulo n, show that

$$\begin{aligned}
\pi(2i+1) - \pi(2i) &= \pi^*(s+5+i) - \pi^*(i) \\
&= (s+5)(3q+3) = n(3q+3)/2 + 3q/2 + 3 \\
&= 3q/2 + 3 = d_{2i} \\
\pi(2i+2) - \pi(2i+1) &= \pi(2i+2) - (\pi(2i) + 3q/2 + 3) \\
&= \pi^*(i+1) - \pi^*(i) - 3q/2 - 3 \\
&= (3q+3) - 3q/2 - 3 = 3q/2 = d_{2i+1}.
\end{aligned}$$

Furthermore, using similar computations as above to verify the few remaining cases, one can show that $\pi(i+1) = \pi(i) + d_i \pmod n$ for all $i = 0, 1, \ldots, n-1$, that is, π

is indeed the associated function (we leave the details to the reader for the sake of brevity).

For $i = 0, 1, \ldots, n-2$, let $f_i = d - d_i + 2$ or more specifically: $f_{2j} = q/2$, and $f_{2j+1} = q/2 + 3$ for $j = 0, 1, \ldots, (n-7)/2$; and $f_{n-2} = f_{n-4} = 3$, and $f_{n-3} = f_{n-5} = q$. Observe that $\mathrm{d}(x_i, x_{i+1}) = \mathrm{d}(v_{\pi(i)}, v_{\pi(i+1)}) = \min\{d_i, n - d_i\} = d_i$ (note that the last equality follows because $d_i \leq d$ when $q \geq 4$) so $f_i = d - \mathrm{d}(x_i, x_{i+1}) + 2$ and item (i) in Lemma 3.1 is satisfied. By inspection,

$$\mathrm{d}(x_i, x_{i+2}) = \mathrm{d}(v_{\pi(i)}, v_{\pi(i_2)})$$
$$= \begin{cases} \min\{3q+3, n-(3q+3)\} = q & \text{if } i \neq n-6, \\ \min\{5q/2+3, n-(5q/2+3)\} = 3q/2 & \text{if } i = n-6, \end{cases}$$

and

$$f_i + f_{i+1} = \begin{cases} q+3 & \text{if } i \neq n-6, \\ 3q/2+3 & \text{if } i = n-6. \end{cases}$$

Therefore, $f_i + f_{i+1} \geq d - \mathrm{d}(x_i, x_{i+2}) + 2$ so item (ii) in Lemma 3.1 is also satisfied. We can conclude that the associated function f is a near-radio labeling of C_n. The span of f is $f(x_{n-1}) = \phi(n)(n-1)/2$ so the desired result follows from item (iv) of Corollary 2.7. □

The case $n = 4q + 3$ where q is a positive multiple of 3 is more complex since the last third of the sequence of integers d_i has descriptions that are significantly different from the first two thirds.

Proposition 3.8. *If $n = 4q + 3$ where q is a positive multiple of* 3, *then*

$$rn^*(C_n) = \phi(n)(n-1)/2 + 1.$$

Proof. Let $s = 8q/3 + 1$ and for $i = 0, 1, \ldots, s$,

$$d_i = \begin{cases} 2q+1 & \text{if } i \text{ even}, \\ q+2 & \text{if } i \text{ odd and } i \neq s, \\ q+1 & \text{if } i = s, \end{cases}$$

and for $j = 0, 1, \ldots, q/3 - 1$,

$$d_{(s+1)+4j} = d_{(s+1)+4j+2} = 2q - 3j$$
$$d_{(s+1)+4j+1} = d_{(s+1)+4j+3} = q + 3 + 3j.$$

Observe that for $j = 0, 1, \ldots, (s-1)/2$, the associated function π is equivalent to

$$\pi(2j) = j(3q+3) \pmod{n} = -jq \pmod{n}$$
$$\pi(2j+1) = j(3q+3) + 2q + 1 \pmod{n} = (2-j)q + 1 \pmod{n}.$$

We will show that π is a permutation of $\{0, 1, \ldots, n-1\}$ is three steps:

Step 1: Let us first show that the set $A = \{\pi(i) : \ i = 0, 1, \ldots, s\}$ has $s + 1$ elements. Suppose to the contrary that this is not true. Therefore, there are two distinct non-negative integers j and j' both not exceeding $(s - 1)/2 = 4q/3$ so that one of the two cases below must hold

- $\pi(2j) = \pi(2j')$ or $\pi(2j + 1) = \pi(2j' + 1)$: From the definition, $(j' - j)q \equiv 0 \pmod n$. Because q and n are both multiples of 3, the last congruence implies $(j' - j)q/3 \equiv 0 \pmod{n/3}$. Therefore, since $\gcd(q/3, n/3) = 1$, we must have $(j' - j) \equiv 0 \pmod{n/3}$, but this is impossible as $0 < |j' - j| \leq 4q/3 < n/3$.
- $\pi(2j) = \pi(2j' + 1)$: From the definition, $(j' - j - 2)q + 1 \equiv 0 \pmod n$. Because q and n are both multiples of 3, the last congruence implies 1 is a multiple of 3, which is also impossible (note that if $a + b \equiv 0 \pmod c$ and m divides both a and c, then m must also divide b).

We reached contradictions in both cases, so we conclude that $|A| = s + 1$.

Step 2: Next, we will show that $\pi(s + 1)$ does not belong to A. Suppose for contradiction that it does and set $j' = (s + 1)/2 = 4q/3 + 1 = n/3$. Therefore, there exists an integer $0 \leq j \leq (s - 1)/2 = 4q/3$ distinct from j' so that $\pi(2j') = \pi(2j)$ or $\pi(2j') = \pi(2j + 1)$. By definition, $\pi(s + 1) = \pi(s) + d_s \pmod n$, hence

$$\begin{aligned}\pi(2j') &= \pi(2(j' - 1) + 1) + (q + 1) \pmod n\\ &= [(j' - 1)(3q + 3) + 2q + 1] + (q + 1) \pmod n\\ &= j'(3q + 3) - 1 \pmod n = -j'q - 1 \pmod n.\end{aligned}$$

Then the equalities $\pi(2j') = \pi(2j)$ or $\pi(2j') = \pi(2j + 1)$ will imply $(j' - j)q + 1 \equiv 0 \pmod n$ or $(j' - j + 2)q + 2 \equiv 0 \pmod n$, respectively. Since q and n are both multiples of 3, the former congruence implies 1 is a multiple of 3, and the latter one implies that 2 is multiple of 3, both impossible, so $\pi(s + 1)$ does not belong to A.

Step 3: Let $A^* = \{0, 1, \ldots, n - 1\} - A$. The objective is to show that $A^* = \{\pi^*(s + i) : \ i = 1, 2, \ldots, n - s - 1\}$ which, together with Steps 1 and 2, allows us to conclude that π is a permutation of $\{0, 1, \ldots, n - 1\}$. We will first show that A^* coincides with the set $B = \{2 + 3i : \ i = 0, 1, \ldots, 4q/3\}$. We have $|B| = 4q/3 + 1 = n - s - 1 = |A^*|$. Therefore, to verify that $A^* = B$, it is enough to show that every element in B does not belong to A. Suppose this is not true, that is, there are non-negative integers i and j not exceeding $4q/3$ such that $\pi(2j) = 2 + 3i$ or $\pi(2j + 1) = 2 + 3i$ which imply $[3i + jq] + 2 \equiv 0 \pmod n$ or $[3i - (2 - j)q] + 1 \equiv 0 \pmod n$, respectively. Since q and n are both multiples of 3, the former congruence implies 2 is a multiple of 3, and the latter implies 1 is a multiple of 3, both impossible. Therefore $A^* = B$. By defining $n^* = |A^*|$ and $q^* = q/3$, we have $n^* = 4q^* + 1$. Consider the auxiliary function π^* on $\{0, 1, \ldots, n^* - 1\}$ such that $\pi^*(0) = 0$ and $\pi^*(i + 1) = \pi^*(i) + d^*_i \pmod{n^*}$ for $i = 0, 1, \ldots, n^* - 2$, where $d^*_i = d_{(s+1)+i}/3$ for $i = 0, 1, \ldots, n^* - 2$, or equivalently, for $j = 0, 1, \ldots, q^* - 1$,

$$d^*_{4j} = d^*_{4j+2} = (2q - 3j)/3 = 2q^* - j$$
$$d^*_{4j+1} = d^*_{4j+3} = (q + 3 + 3j)/3 = q^* + 1 + j.$$

We previously argued in the proof of Proposition 3.3 that π^* is a permutation of $\{0, 1, \ldots, n^* - 1\}$. From Step 2, we have $\pi(s+1)$ in A^*, thus let l be the integer so that $\pi(s+1) = 2+3l$ and consider the isomorphism h between sets $\{0, 1, \ldots, 4q/3\}$ and A^* such that $h(i) = 2 + 3(l + i) \pmod n$. Since $\pi(s + i) = h(\pi^*(i - 1))$ for $i = 1, 2, \ldots, n^*$, we can conclude $A^* = \{\pi(s + i) : \; i = 1, 2, \ldots, n - s - 1\}$.

For $i = 0, 1, \ldots, n - 2$, let $f_i = d - d_i + 2$ or more specifically for $i = 0, 1, \ldots, s$,

$$f_i = \begin{cases} 2 & \text{if } i \text{ even}, \\ q+1 & \text{if } i \text{ odd and } i \neq s, \\ q+2 & \text{if } i = s, \end{cases}$$

and for $j = 0, 1, \ldots, q/3 - 1$,

$$f_{(s+1)+4j} = f_{(s+1)+4j+2} = 3 + 3j$$
$$f_{(s+1)+4j+1} = f_{(s+1)+4j+3} = q - 3j.$$

Item (i) in Lemma 3.1 is trivially satisfied as $\mathrm{d}(x_i, x_{i+1}) = \mathrm{d}(v_{\pi(i)}, v_{\pi(i+1)}) = \min\{d_i, n - d_i\} = d_i$ (note that the last equality follows because $d_i \leq d$). By inspection, we have for $i = 0, 1, \ldots, n - 3$:

$$f_i + f_{i+1} = \begin{cases} q+3 & \text{if } (i \leq s-2) \text{ or } (i \geq s+1 \text{ and } i - s \text{ not a multiple of } 4), \\ q+4 & \text{if } i = s-1, \\ q+5 & \text{if } i = s, \\ q+6 & \text{otherwise}, \end{cases}$$

$$d_i + d_{i+1} = \begin{cases} 3q+3 & \text{if } (i \leq s-2) \text{ or } (i \geq s+1 \text{ and } i - s \text{ not a multiple of } 4), \\ 3q+2 & \text{if } i = s-1, \\ 3q+1 & \text{if } i = s, \\ 3q & \text{otherwise}. \end{cases}$$

Hence $d < 3q < d_i + d_{i+1} \leq 3q + 3 < n$, and we must have $\mathrm{d}(x_i, x_{i+2}) = n - (d_i + d_{i+1})$. Then

$$f_i + f_{i+1} \geq q + 3 \geq (3q + 3) - 2q \geq (d_i + d_{i+1}) - 2q = d - \mathrm{d}(x_i, x_{i+2}) + 2.$$

Thus, item (ii) in Lemma 3.1 is also satisfied, and we can conclude that the associated function f is a near-radio labeling of C_n with span $f(x_{n-1}) = \phi(n)(n-1)/2+1$. The proposition follows from item (iv) of Corollary 2.7. □

4. Closing Remarks

We provide non-trivial lower bounds for the radio k-chromatic numbers of cycles with $n \geq 3$ vertices for all k at least as large as the diameter $d = \lfloor n/2 \rfloor$. These lower bounds coincide with the exact values when $k = d$ as shown in Liu and Zhu [13]. We could also confirm our lower bounds are exact when $k = d + 1$, but exhibiting radio k-labelings with spans achieving these bounds was considerably challenging in some instances. We conjecture that similar techniques could also be used to find exact radio k-chromatic numbers of cycles for other $k > d + 1$, but they may not be straightforward extensions of the ones used for the case $k = d + 1$. The lower bounds' dependence on the relationship between k and n makes it unlikely that a general set of labeling schemes could achieve the radio k-chromatic number for different k.

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