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The minimum span of $L(2, 1)$-labelings of generalized flowers

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ABSTRACT

Given a positive integer d, an $L(d, 1)$-*labeling* of a graph G is an assignment of nonnegative integers to its vertices such that adjacent vertices must receive integers at least d apart, and vertices at distance two must receive integers at least one apart. The λ_d-number of G is the minimum k so that G has an $L(d, 1)$-labeling using labels in $\{0, 1, \ldots, k\}$. Informally, an *amalgamation* of two disjoint graphs G_1 and G_2 along a fixed graph G_0 is the simple graph obtained by identifying the vertices of two induced subgraphs isomorphic to G_0, one in G_1 and the other in G_2. A *flower* is an amalgamation of two or more cycles along a single vertex. We provide the exact λ_2-number of a *generalized flower* which is the Cartesian product of a path P_n and a flower, or equivalently, an amalgamation of cylindrical rectangular grids along a certain P_n. In the process, we provide general upper bounds for the λ_d-number of the Cartesian product of P_n and any graph G, using *circular* $L(d+1, 1)$-labelings of G where the labels $\{0, 1, \ldots, k\}$ are arranged sequentially in a circle and the distance between two labels is the shortest distance on the circle.

1. Introduction

An $L(2, 1)$-*labeling* of a graph G is a function $f : V(G) \to \{0, 1, \ldots, k\}$ such that if vertices u and v are adjacent, then $|f(u) - f(v)| \geq 2$, and if u and v are at distance two, then $|f(u) - f(v)| \geq 1$. An $L(2, 1)$-labeling that uses labels in the set $\{0, 1, \ldots, k\}$ will be called a k-*labeling* of G. The minimum k so that G has a k-labeling will be called the λ-*number* of G and denoted by $\lambda(G)$. The study of $L(2, 1)$-labelings and their variations is motivated by the channel assignment problem [11] and has generated a vast literature since $L(2, 1)$-labelings were first introduced in 1992 [10]. We refer the reader to the surveys in [2,28] for an overview and to [3–5,15,16,20,21,23,25,29] for some of the more recent developments in the field.

The long-standing conjecture in the field states that $\lambda(G) \leq \Delta^2(G)$, where $\Delta(G)$ denotes the maximum degree of G [10]. This conjecture holds for graphs with $\Delta(G)$ larger than approximately 10^{69} [12] and for graphs with at most $(\lfloor \Delta(G)/2 \rfloor + 1)(\Delta^2(G) - \Delta(G) + 1) - 1$ vertices [5]. The best known general upper bound is $\lambda(G) \leq \Delta^2(G) + \Delta(G) - 2$ [9]. Even though the problem of determining $\lambda(G)$ is NP-hard [8], several bounds and exact λ-numbers for different families of graphs are known. One of these families is the class of amalgamations of graphs studied in [1].

Let $G_1, G_2, \ldots, G_p$ be $p \geq 2$ pairwise disjoint graphs each containing a fixed induced subgraph isomorphic to a graph G_0. The *amalgamation* of $G_1, G_2, \ldots, G_p$ *along* G_0 is the simple graph $G = Amalg(G_0; G_1, G_2, \ldots, G_p)$ obtained by identifying $G_1, G_2, \ldots, G_p$ at the vertices in the fixed subgraphs isomorphic to G_0 in each $G_1, G_2, \ldots, G_p$, respectively. In [1], general

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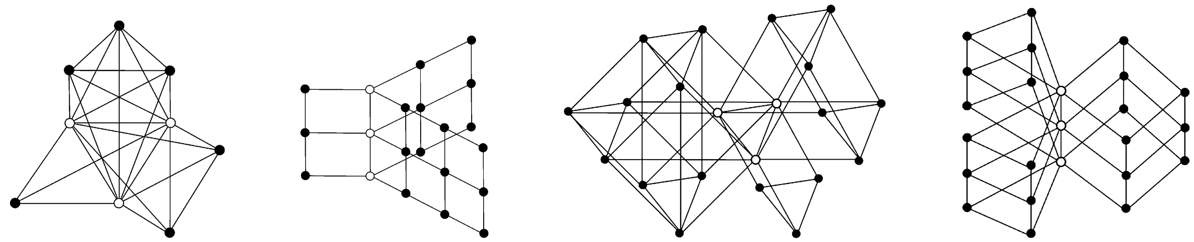


(a) $Amalg(K_3; K_6, K_5, K_4)$. (b) $Amalg(P_3; P_3 \square P_4, P_3 \square P_3, P_3 \square P_2)$. (c) $Amalg(K_3; K_3 \square K_4, K_3 \square K_3, K_3 \square K_2)$. (d) $Amalg(P_3; P_3 \square C_4, P_3 \square C_3, P_3 \square C_3)$.

Fig. 1.1. Examples[1] of $Amalg(G_0; G_1, G_2, \ldots, G_p)$ where the white vertices are in G_0.

upper bounds for the λ-number of the amalgamation of graphs were established by determining the exact λ-number of the amalgamation of complete graphs along a complete graph (see Fig. 1.1(a)). They also provided the exact λ-numbers of the amalgamation of *rectangular grids* (i.e., of Cartesian products of two paths) along a certain path, or more specifically, of the Cartesian product of a path and a star with spokes of arbitrary lengths (see Fig. 1.1(b)).

Recall that the *Cartesian product* of two disjoint graphs H_1 and H_2, denoted by $H_1 \square H_2$, is defined as the graph with vertex set given by the Cartesian product of the vertex set of H_1 and the vertex set of H_2, where two vertices (u, v) and (w, z) are adjacent if and only if either (u, w are adjacent in H_1 and $v = z$) or (v, z are adjacent in H_2 and $u = w$). The λ-numbers of the Cartesian products of graphs as simple as paths P_n, cycles C_n, and complete graphs K_n (where n is the number of vertices in the respective graphs), have been extensively investigated in the literature, often generating challenging problems, some of them still open to date. For a sample of related works, we refer the reader to [7,14,17,18,24,26].

Expanding upon other amalgamations of Cartesian products of graphs, [15] provided a tight upper bound for the λ-number of amalgamations of Cartesian products of two complete graphs along a complete graph (see Fig. 1.1(c)) and the exact λ-numbers for certain infinite subclasses of amalgamations of this form. The complete determination of this λ-number may be out of reach as indicated in [15] due to a surprising relationship between this problem and the NP-hard minimum makespan scheduling problem [6].

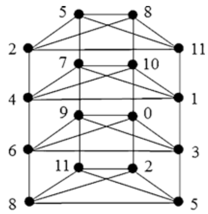
In this work, we will focus on the amalgamation G of *cylindrical rectangular grids* (i.e., of Cartesian products of a path and a cycle) along a fixed path, or more formally, for given integers $n \geq 1, p \geq 2$, and $3 \leq m_1 \leq m_2 \leq \cdots \leq m_p$, $G = Amalg(G_0; G_1, G_2, \ldots, G_p)$ with $G_0 = P_n$ and $G_\ell = P_n \square C_{m_\ell}$ for $\ell = 1, 2, \ldots, p$. We will call G a *generalized flower* and denote it by $F_n(m_1, m_2, \ldots, m_p)$ (see $F_3(3, 3, 4)$ in Fig. 1.1(d)). When $n = 1$, a generalized flower is the amalgamation of cycles along a single vertex which we will simply call a *flower* for the obvious resemblance to its counterpart in nature. We extend this metaphor to all the generalized flowers and refer to the graph $G_0 = P_n$ as the *stem*, and the graph $G_\ell = P_n \square C_{m_\ell}$ as *petal* ℓ, for $\ell = 1, 2, \ldots, p$. Our ultimate goal will be to show the following:

Theorem 1.1. *Let $n \geq 1, p \geq 2, 3 \leq m_1 \leq m_2 \leq \cdots \leq m_p$ be integers and let $G = F_n(m_1, m_2, \ldots, m_p)$. Then*

$$\begin{aligned}
\lambda(G) &= 2p+1, && \textit{if } n = 1;\\
&= 2p+2, && \textit{if } (n = 2 \textit{ and } p > 2) \textit{ or}\\
& && (n = 2, p = 2, (m_1, m_2) \notin \{(4, 4), (4, 8)\}, \textit{ and } \{m_1, m_2\} \cap \{3, 6\} = \varnothing);\\
&= 2p+4, && \textit{if } (n = 3, p = 2, \textit{ and } (m_1, m_2) \in \{(3, 3), (3, 5)\}) \textit{ or}\\
& && (n = 4, p = 2, \textit{ and } (m_1, m_2) \in \{(3, 3), (3, 5), (3, 6), (4, 4), (4, 5)\});\\
&= 2p+4, && \textit{if } n \geq 5;\\
&= 2p+3, && \textit{otherwise}.
\end{aligned}$$

The following two well-known generalizations of $L(2, 1)$-labelings will be used in Section 2. Let d and k be integers such that $d \geq 1$ and $k \geq 0$. An *$L(d, 1)$-labeling* of a graph G is a function $f : V(G) \to \{0, 1, \ldots, k\}$ such that if vertices u and v are adjacent, then $|f(u) - f(v)| \geq d$, and if u and v are at distance two, then $|f(u) - f(v)| \geq 1$. An $L(d, 1)$-labeling that uses labels in the set $\{0, 1, \ldots, k\}$ will be called a *(k, d)-labeling* of G. The minimum k so that G has a (k, d)-labeling will be called the *λ_d-number* of G and denoted by $\lambda_d(G)$. A *k-circular $L(d, 1)$-labeling* of a graph G is a function $f : V(G) \to \{0, 1, \ldots, k\}$ such that if vertices u and v are adjacent, then $\|f(u) - f(v)\|_k \geq d$, and if u and v are at distance two, then $\|f(u) - f(v)\|_k \geq 1$, where $\|x\|_k = \min\{|x|, (k + 1) - |x|\}$. (Note: The traditional definition of a k-circular $L(d, 1)$-labeling of G in the literature maps $V(G)$ into $\{0, 1, \ldots, k - 1\}$ but we chose to adapt it to be compatible with the definition of a (k, d)-labeling which maps $V(G)$ into $\{0, 1, \ldots, k\}$.) For more on this labeling variation, we refer the interested reader to [13,19,22,27]. It is helpful to label the vertices of C_{k+1} clockwise around the cycle with $0, 1, \ldots, k$, and interpret $\|x - y\|_k$, called the *k-circular distance*

1 Fig. 1.1(a) and (b) are from [1], and (c) is from [15].



$$A = \begin{array}{|cccc|} \hline 2 & 5 & 8 & 11 \\ 4 & 7 & 10 & 1 \\ 6 & 9 & 0 & 3 \\ 8 & 11 & 2 & 5 \\ \hline \end{array}$$

Fig. 2.1. $(11, 2)$-labeling of $P_4 \,\square\, K_4$ constructed in the proof of Theorem 2.1.

between the labels x and y, as the shortest distance between x and y on the cycle. Note that a k-circular $L(d, 1)$-labeling is also a (k, d')-labeling for any positive integer $d' \leq d$ but the converse is not true in general.

In Section 2, we use k-circular $L(d+1, 1)$-labelings to provide upper bounds for the λ_d-number of the Cartesian products of a path and any graph. This result will be helpful in obtaining the general upper bound $2p+4$ for the λ-number of generalized flowers; this bound is exact when $n \geq 5$. We would like to highlight that the strength of this result lies in the fact that it provides a detailed framework to confirm the general upper bounds for the λ-number of the Cartesian products of two paths [26], of a path and a cycle [14], and of a path and a star with spokes of arbitrary lengths [19]. These upper bounds were originally found using methods that either did not use circular labelings or did not acknowledge their role in obtaining the desired λ-labelings.

When determining the exact λ-numbers of families of graphs, the instances with smaller number of vertices are often discussed separately as the proximity among vertices may require a more involved selection of labels in $L(2, 1)$-labelings achieving the λ-numbers. This selection of labels may also be very sensitive to the structure of each particular instance, adding to the complexity of the determination problem. Sections 3 through 5 discuss the smaller instances of generalized flowers, namely when $n \leq 4$. The λ-number of flowers, i.e., of generalized flowers with $n = 1$, is given in Section 3. The case $n = 2$ is treated in Section 4, and the cases $n = 3, 4$ are treated in Section 5.

2. General upper bound for the λ-number of generalized flowers and an exact value when $n \geq 5$

In Theorem 2.4, we exhibit a general upper bound for the λ-number of $F_n(m_1, m_2, \ldots, m_p)$ and use it in Corollary 2.6 to determine the exact λ-number when $n \geq 5$. We first need to prove Theorem 2.1 and Corollary 2.2 that use circular $L(d+1, 1)$-labelings to derive an upper bound for the λ_d-number of the Cartesian product of a path and any graph.

Theorem 2.1. *Let d and k be integers such that $d \geq 1$ and $k \geq 2d$. If a graph G has a k-circular $L(d+1, 1)$-labeling, then $P_n \,\square\, G$ has a (k, d)-labeling.*

Proof. Let $u_0, u_1, \ldots, u_{n-1}$ be the vertices in P_n so that u_i and u_{i+1} are adjacent for $i = 0, 1, \ldots, n-2$, and let $v_0, v_1, \ldots, v_{m-1}$ be the vertices in G. Let g be a k-circular $L(d+1, 1)$-labeling of G. We will construct a (k, d)-labeling f of $P_n \,\square\, G$ as an n-by-m matrix A where the entry $A_{i,j}$ on the ith row, jth column will be the label $f(u_i, v_j)$ of vertex (u_i, v_j) for $i = 0, 1, \ldots, n-1$ and $j = 0, 1, \ldots, m-1$. For each $j = 0, 1, \ldots, m-1$, set $A_{0,j} = g(v_j)$ and $A_{i,j} = (A_{i-1,j} + d) \bmod (k+1)$ for $i = 1, 2, \ldots, n-1$. Fig. 2.1 illustrates this construction when $d = 2$, $k = 11$, $n = 4$, and $G = K_4$, the complete graph on 4 vertices.

We make the following two claims:

(i) Each row of A induces a k-circular $L(d+1, 1)$-labeling of G if the jth label in the row is assigned to v_j for $j = 0, 1, \ldots, m-1$. This holds since the 0th row of A is g and each of the remaining rows is a circular shift of the 0th row by a multiple of d (note that the k-circular distance between the labels assigned to an arbitrary pair of vertices in G remains constant through these circular shifts). Consequently, each row of A induces a (k, d)-labeling of G.

(ii) Each column of A induces a k-circular $L(d, 1)$-labeling of P_n if the ith label in the column is assigned to u_i for $i = 0, 1, \ldots, n-1$. To verify this statement, recall that $k \geq 2d$ and each entry in a column is obtained from the previous one in the same column by adding d modulo $k+1$. Hence, the k-circular distance between the labels assigned to two adjacent vertices in P_n is exactly d (since $d < (k+1) - d$). In addition, the labels assigned to two vertices at distance two in P_n must be different (since one of these labels is obtained from the other by adding $2d$ modulo $k+1$). Thus, the k-circular distance between these labels is at least 1. As in item (i), we can finally conclude that each column of A induces a (k, d)-labeling of P_n.

In view of these two claims, to conclude that f is a (k, d)-labeling of $P_n \,\square\, G$, it is sufficient to show that if there exist two adjacent vertices u and u' in P_n and two adjacent vertices v and v' in G, i.e., (u', v) and (u, v') are both adjacent to (u, v) in $P_n \,\square\, G$, then $|f(u', v) - f(u, v')| \geq 1$. To simplify the notation, we call $a = f(u', v)$, $b = f(u, v')$, and $c = f(u, v)$. By our construction, we have $\|b - c\|_k \geq d + 1$ since v and v' are adjacent in G, and $\|a - c\|_k = d$ since u and u' are adjacent in P_n, and would like to show that $a \neq b$. If $a = b$, then $\|b - c\|_k \geq d + 1$ implies $\|a - c\|_k \geq d + 1$ which contradicts $\|a - c\|_k = d$. Therefore $a \neq b$ as desired and we conclude that f is a (k, d)-labeling of $P_n \,\square\, G$. $\square$

It is important to note that the k-circular requirement for the $L(d+1,1)$-labeling g of G in the proof of Theorem 2.1 is essential in guaranteeing that the final labeling is indeed a (k,d)-labeling of $P_n \square G$. For example, if this requirement is dropped when $k = 8$ and $d = 2$, then two adjacent vertices in G could be labeled 0 and 7, respectively. Adding $d = 2$ modulo $k+1 = 9$ to each label, we obtain 2 and 0, respectively. This is not possible in an $L(2,1)$-labeling, as there are two vertices at distance two with the same label, namely 0. More generally, by dropping the k-circular requirement for the $L(d+1,1)$-labeling, the pairs $(i, k-j)$ for $i = 0,1,\ldots,d-1$ and $j = 0,1,\ldots,d-i-1$ would be possible pairs of labels for two adjacent vertices in G, respectively, since $k \geq 2d$ implies $(k-j)-i \geq d+1$. Adding d modulo $k+1$ to each label in these pairs we would obtain the pairs $(d+i, d-j-1)$ for $i = 0,1,\ldots,d-1$ and $j = 0,1,\ldots,d-i-1$ assigned to adjacent vertices in $P_n \square G$. This is not possible in a (k,d)-labeling because if $j < d-i-1$ then we will have two adjacent vertices with labels differing by $(d+i)-(d-j-1) \leq d-1$, and if $j = d-i-1$ we will have two vertices at distance two with the same label i.

Theorem 2.1 confirms several of the known general upper bounds for the λ-number of $P_n \square G$ for different families of graphs G. For example, it was shown in [19] that P_m has a 6-circular $L(3,1)$-labeling, so Theorem 2.1 implies that there exists a 6-labeling of $P_n \square P_m$, confirming the upper bound $\lambda(P_n \square P_m) \leq 6$ found in [26]. Similarly, it was shown in [22] that C_m has a 7-circular $L(3,1)$-labeling and therefore there exists a 7-labeling of $P_n \square C_m$ confirming that $\lambda(P_n \square C_m) \leq 7$ as shown in [14]. Also from [19], there exists a $(q+4)$-circular $L(3,1)$-labeling of a tree T with one vertex with maximum degree $q > 2$ and all the other vertices with degree at most 2, and therefore there exists a $(q+4)$-labeling of $P_n \square T$ confirming that $\lambda(P_n \square T) \leq q+4$ as shown in [1]. We also have the following corollary that provides a general upper bound for the λ_d-number of the Cartesian product of a path and any graph.

Corollary 2.2. *If d is a positive integer, then $\lambda_d(P_n \square G) \leq \lambda_{d+1}(G) + d$ for any graph G.*

Proof. As observed in [13,22], we have that the minimum k so that G has a k-circular $L(d+1,1)$-labeling is at most $\lambda_{d+1}(G) + d$, and therefore the desired inequality follows from Theorem 2.1. □

Before proceeding to the next results, we need to introduce some notation. Let $G = F_n(m_1, m_2, \ldots, m_p)$ be a generalized flower where $n \geq 1$, $p \geq 2$, and $3 \leq m_1 \leq m_2 \leq \cdots \leq m_p$. For each $\ell = 1,2,\ldots,p$, the petal ℓ of G is isomorphic to $P_n \square C_{m_\ell}$ so its vertices can be organized in an array format where each vertex will be represented by an ordered triple (i,j,ℓ) with $i = 0,1,\ldots,n-1$ and $j = 0,1,\ldots,m_\ell - 1$, where the 0th column contains the vertices in the stem P_n. Two vertices are adjacent if their triple representations satisfy exactly one of the following two conditions:

(i) Both triples agree on the first and third coordinates, and differ in absolute value by 1 or by $m_\ell - 1$ on the second coordinate.
(ii) Both triples agree on the second and third coordinates, and differ in absolute value by 1 on the first coordinate.

For a fixed $i = 0,1,\ldots,n-1$, the subgraph of G induced by the vertices (i,j,ℓ) with $j = 0,1,\ldots,m_\ell - 1$ and $\ell = 1,2,\ldots,p$ is isomorphic to $F_1(m_1, m_2, \ldots, m_p)$ and will be called the ith *layer of* G. If, in addition to fixing i, we also fix $\ell = 1,2,\ldots,p$, the subgraph of the ith layer of G induced by the vertices (i,j,ℓ) with $j = 0,1,\ldots,m_\ell - 1$ is isomorphic to C_{m_ℓ} and will be called the ith *layer of petal* ℓ. For convenience, the $L(2,1)$-labelings of $F_n(m_1, m_2, \ldots, m_p)$ will be represented by the n-by-m_ℓ matrices for $\ell = 1,2,\ldots,p$, where the entry on the ith row, jth column of the ℓth matrix will be the label of vertex (i,j,ℓ); observe that all the 0th columns of these p matrices must be the same as they contain the labels for the stem P_n of G.

In the proof of Theorem 2.4, we will first treat the case $F_n(3,3)$ separately, and for each of the remaining generalized flowers $F_n(m_1, m_2, \ldots, m_p)$, we will construct a $(2p+4)$-circular $L(3,1)$-labeling of the 0th layer and then use Theorem 2.1 to obtain a $(2p+4)$-labeling of the entire graph. In such a construction, we will need k-circular $L(3,1)$-labelings of single cycles C_m with $m \geq 4$ satisfying the special properties stated in the following result.

Lemma 2.3. *Let k and m be integers such that $k \geq 8$ and $m \geq 4$. For each $x = 3,4,\ldots,k-3$, there exists a k-circular $L(3,1)$-labeling of C_m which assigns label 0 to an arbitrary vertex and labels x and $x+1$ to the vertices adjacent to it.*

Proof. Let $v_0, v_1, \ldots, v_{m-1}$ be the vertices in C_m so that v_i and v_{i+1} are adjacent for $i = 0,1,\ldots,m-1$ and subscript addition is taken modulo m. In Fig. 2.2 we exhibit the desired k-circular $L(3,1)$-labelings of C_m as row-vectors of labels where the ith entry in each vector contains the label of vertex v_i for $i = 0,1,\ldots,m-1$. Each shaded block of three consecutive labels within a vector is a k-circular $L(3,1)$-labeling of C_3 that can be replaced with $q \geq 1$ copies of itself arranged consecutively as needed to reach the desired value of m. For example, if $x = 4$ and $m = 8$, then $m = 2 + 3q$ for $q = 2$ so the corresponding vector in Fig. 2.2 produces the circular labeling $0, \underline{4,7,1}, \underline{4,7,1}, 5$ when read off clockwise around C_8 starting at v_0. We leave the verification that these are indeed k-circular $L(3,1)$-labelings of C_m to the reader but would like to note that since $k \geq 8$ and $x = 3,4,\ldots,k-3$, the pairs $(0,k)$, $(0,k-1)$, $(1,k)$ are not used to label two adjacent vertices. □

Theorem 2.4. *Let $n \geq 1$, $p \geq 2$, $3 \leq m_1 \leq m_2 \leq \cdots \leq m_p$ be integers. If $G = F_n(m_1, m_2, \ldots, m_p)$, then $\lambda(G) \leq 2p+4$.*

Proof. Let $s = 0$ if $m_1 > 3$ or let s be the largest positive integer such that $m_s = 3$.

If $s = p = 2$, let t be the smallest multiple of 3 so that $t \geq n$. Use $t/3$ copies of matrix A in Fig. 2.3 arranged vertically to label one of the petals of $F_t(3,3)$. Similarly label the other petal of $F_t(3,3)$ using matrix B in Fig. 2.3.

$q \geq 1$	$x = 3$	$x = 4$	$x = 5$
C_4	0 3 7 4	0 4 8 5	0 5 2 6
C_{2+3q}	0 3 6 1 4	0 4 7 1 5	0 5 8 2 6
C_{3+3q}	0 3 6 1 7 4	0 4 1 7 2 5	0 5 2 8 3 6
C_{4+3q}	0 3 6 2 5 8 4	0 4 1 7 2 8 5	0 5 2 8 5 2 6

$q \geq 1$	$6 \leq x \leq k - 3$
C_{1+3q}	0 x 3 $x + 1$
C_{2+3q}	0 x 1 4 $x + 1$
C_{3+3q}	0 x 3 0 4 $x + 1$

Fig. 2.2. k-circular $L(3, 1)$-labeling of C_m with $m \geq 4$ (shaded blocks are repeated $q \geq 1$ times).

$$A = \begin{array}{|ccc|} \hline 0 & 2 & 4 \\ 3 & 5 & 7 \\ 6 & 8 & 1 \\ \hline \end{array} \qquad B = \begin{array}{|ccc|} \hline 0 & 5 & 7 \\ 3 & 8 & 1 \\ 6 & 2 & 4 \\ \hline \end{array}$$

Fig. 2.3. Matrices A and B used construct an 8-labeling of $F_t(3, 3)$.

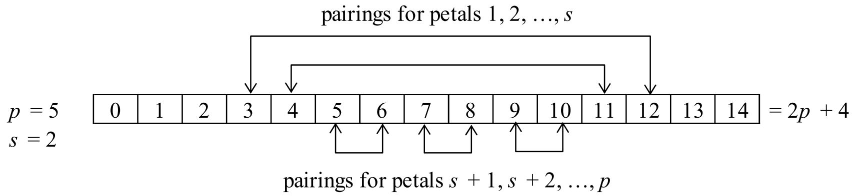


Fig. 2.4. Pairs of labels used for the vertices adjacent to the vertex in the stem on the 0th layer of $G = F_n(m_1, m_2, \ldots, m_p)$ when $p = 5$ and $s = 2$ in Theorem 2.4.

petal 1				**petal 2**				**petal 3**					**petal 4**					**petal 5**					
	0	3	12		0	4	11		0	5	2	6		0	7	3	8		0	9	1	4	10
	2	5	14		2	6	13		2	7	4	8		2	9	5	10		2	11	3	6	12
	4	7	1		4	8	0		4	9	6	10		4	11	7	12		4	13	5	8	14
	6	9	3		6	10	2		6	11	8	12		6	13	9	14		6	0	7	10	1
	8	11	5		8	12	4		8	13	10	14		8	0	11	1		8	2	9	12	3

Fig. 2.5. $L(2, 1)$-labeling of $G = F_5(3, 3, 4, 4, 5)$ constructed in Theorem 2.4.

By inspection, this is an 8-labeling of $F_t(3, 3)$ and therefore $\lambda(G) \leq 8 = 2p + 4$ since G is a subgraph of $F_t(3, 3)$.

If $s = p > 2$, label the 0th layer of petal i with labels $0, i + 2, i + p + 2$ for $i = 1, 2, \ldots, p$ where the label 0 is assigned to the vertex in the stem. These are all $(2p + 4)$-circular $L(3, 1)$-labelings of C_3 and together induce a $(2p + 4)$-circular $L(3, 1)$-labeling of $F_1(3, 3, \ldots, 3)$ with p petals which is isomorphic to the 0th layer of G. By Theorem 2.1, there exists a $(2p + 4)$-labeling of $P_n \square F_1(3, 3, \ldots, 3)$ with p petals, which is isomorphic to G. Thus, $\lambda(G) \leq 2p + 4$.

We can finally consider the case where $0 \leq s < p$. Similarly to the previous case, it is sufficient to exhibit a $(2p + 4)$-circular $L(3, 1)$-labeling of the 0th layer of G to conclude that $\lambda(G) \leq 2p + 4$ by Theorem 2.1. For $i = 1, 2, \ldots, s$, label the 0th layer of petal i with labels $0, i + 2, 2p - i + 3$ where the label 0 is assigned to the vertex in the stem. These are all $(2p + 4)$-circular $L(3, 1)$-labelings of C_3 which together induce a $(2p + 4)$-circular $L(3, 1)$-labeling of $F_1(3, 3, \ldots, 3)$ with s petals. Extend this labeling to the remaining vertices in the 0th layer of G by labeling the 0th layer of petal $s + j$ for each $j = 1, 2, \ldots, p - s$ with a $(2p + 4)$-circular $L(3, 1)$-labeling of $C_{m_{s+j}}$ which assigns label 0 to the vertex in the stem and labels $s + 2j + 1$ and $s + 2j + 2$ to the vertices adjacent to it. Lemma 2.3 guarantees the existence of such labelings by selecting $k = 2p + 4$ (note that $3 \leq s + 2j + 1 \leq s + 2(p - s) + 1 \leq 2p + 1 = k - 3$). The resulting labeling is a $(2p + 4)$-circular $L(3, 1)$-labeling of the 0th layer of G. We leave the verification of this claim to the reader but the example in Fig. 2.4 might be helpful in understanding the choice of labels for the vertices adjacent to the vertex in the stem; if $n = 5$, the corresponding final $L(2, 1)$-labeling is shown in Fig. 2.5. □

In Corollary 2.6 we show that the upper bound in Theorem 2.4 is tight when $n \geq 5$ using the following well-known result.

Result 2.5 (*[10]*)**.** If a graph G contains three vertices with maximum degree $\Delta(G) \geq 2$ and one of them is adjacent to the other two vertices, then $\lambda(G) \geq \Delta(G) + 2$.

Corollary 2.6. *Let $n \geq 5$, $p \geq 2$, $3 \leq m_1 \leq m_2 \leq \cdots \leq m_p$ be integers. If $G = F_n(m_1, m_2, \ldots, m_p)$, then $\lambda(G) = 2p + 4$.*

Proof. This follows immediately from Theorem 2.4 and Result 2.5 since there are three vertices in the stem with degrees $\Delta(G) = 2p + 2$ and one of them is adjacent to the other two vertices. □

$q \geq 1$	$k = 1$	$k = 2$	$k \geq 3$
C_{1+3q}	0 2 5 3	0 4 2 5	0 $2k$ 2 $2k+1$
C_{2+3q}	0 2 4 1 3	0 4 1 3 5	0 $2k$ 2 4 $2k+1$
C_{3+3q}	0 2 4 1 5 3	0 4 2 0 3 5	0 $2k$ 2 0 4 $2k+1$

Fig. 3.1. $L(2, 1)$-labelings of C_m with $m \geq 4$ in Lemma 3.1 (shaded blocks are repeated $q \geq 1$ times).

$q \geq 1$	$s \geq 2$ and $b = 1$
C_{1+3q}	0 2 4 $2p+1$
C_{2+3q}	0 2 4 1 $2p+1$
C_{3+3q}	0 2 4 0 2 $2p+1$

$q \geq 1$	$s = 1$ and $b = 1$
C_4	0 3 1 4
C_{2+3q}	0 3 5 1 4
C_{3+3q}	0 3 5 0 2 4
C_{4+3q}	0 3 1 5 0 2 4

Fig. 3.2. $(2p+1)$-labelings of C_m with $m \geq 4$ in Theorem 3.2 (shaded blocks are repeated $q \geq 1$ times).

3. The λ-number of flowers ($n = 1$)

It is well known that the λ-number of any graph is at least one more than its maximum degree. So if G is a flower with $p \geq 2$ petals, then its λ-number is between $\Delta(G) + 1 = 2p + 1$ and the general upper bound $2p + 4$ of Theorem 2.4. In this section we show that this lower bound is actually the exact λ-number of G.

Lemma 3.1. *Let k and m be integers such that $k \geq 1$ and $m \geq 4$. Then there exists (a 5-labeling, if $k = 1$) or (a $(2k+1)$-labeling, if $k > 1$) of C_m which assigns label 0 to an arbitrary vertex and labels $2k$ and $2k+1$ to the vertices adjacent to it.*

Proof. As in the proof of Lemma 2.3, the desired labelings are exhibited in Fig. 3.1. □

Theorem 3.2. *Let $p \geq 2$ and $3 \leq m_1 \leq m_2 \leq \cdots \leq m_p$ be integers. If $G = F_1(m_1, m_2, \ldots, m_p)$, then $\lambda(G) = 2p + 1$.*

Proof. Let w be the only vertex in the stem of G and let $s = 0$ if $m_1 > 3$ or let s be the largest positive integer such that $m_s = 3$. Set $b = p - s$. Let H_1 (resp., H_2) be the subgraph of G induced by the vertices in petals $1, 2, \ldots, s$ (resp., $s+1, s+2, \ldots, p$). To show that $\lambda(G) = \Delta(G) + 1 = 2p + 1$, we will construct a $(2p+1)$-labeling f of G using $L(2, 1)$-labelings of H_1 and H_2, respectively, for different combinations of values of s and b. For the sake of brevity, we will omit the formal verification that each f is indeed an $(2p+1)$-labeling as the proofs are fairly straightforward.

If $s \geq 2$, then H_1 is isomorphic to $Amalg(K_1; K_3, K_3, \ldots, K_3)$ with s copies of K_3 and from [1] we have $\lambda(H_1) = 2s + 1$ (a particular case of Theorem 2.3 on p. 883 of [1]; note that this result requires $s \geq 2$); let f_1 be an arbitrary $(2s+1)$-labeling of H_1. Note that we must have $f_1(w) = 0$ and the $2s$ vertices adjacent to w are assigned the different labels $2, 3, \ldots, 2s+1$. Hence, if $b = 0$, then $p = s$ and we can set $f = f_1$.

If $b \geq 2$, then H_2 is isomorphic to $Amalg(P_1; C_{m_{s+1}}, C_{m_{s+2}}, \ldots, C_{m_p})$. Let f_2 be the $(2b+1)$-labeling of H_2 using Lemma 3.1 to label each $C_{m_{s+k}}$ for $k = 1, 2, \ldots, b$ where the label 0 is assigned to w and labels $2k$ and $2k+1$ are assigned to the vertices adjacent to w. Hence, if $s = 0$, then $p = b$ and we can set $f = f_2$.

If $s \geq 2$ and $b \geq 2$, then set $f(v) = f_1(v) + 2b$ if $v \in H_1$ and $v \neq w$, and set $f(v) = f_2(v)$ otherwise.

If $s = 1$ and $b \geq 2$, then set $f(v) = f_2(v) + 1$ if $v \in H_2$ and $v \neq w$, $f(w) = 0$, and if x, y are the two vertices in H_1 (isomorphic to C_3) adjacent to w, set $f(x) = 2$ and $f(y) = 2p + 1$.

If $s \geq 2$ and $b = 1$, then set $f(v) = f_1(v) + 1$ if $v \in H_1$ and $v \neq w$, $f(w) = 0$, and if x, y are the two vertices in H_2 (isomorphic to C_m for some $m \geq 4$) adjacent to w, set $f(x) = 2$ and $f(y) = 2p + 1$. In the left-most table of Fig. 3.2, we exhibit the desired $(2p+1)$-labelings of C_m extending the given partial labeling.

If $s = 1$ and $b = 1$, then set $f(u) = 2$ and $f(v) = 5$ where u, v are the two vertices in H_1 (isomorphic to C_3) adjacent to w, $f(w) = 0$, and if x, y are the two vertices in H_2 (isomorphic to C_m for some $m \geq 4$) adjacent to w, set $f(x) = 3$ and $f(y) = 4$. In the right-most table of Fig. 3.2, we exhibit the desired $(2p+1)$-labelings of C_m extending the given partial labeling. □

4. The λ-number of generalized flowers with $n = 2$

In this section, we relied on a simple backtracking computer program to find the λ-numbers of a finite family of graphs that are subgraphs of certain generalized flowers with $n = 2$. The same program was also used to construct $L(2, 1)$-labelings for individual petals of a given generalized flower with $n = 2$ that together provided an $L(2, 1)$-labeling for the entire graph. We will not present formal verifications of these facts as they are long and tedious case discussions but will exhibit the labelings we used so that the interested reader can check that they are indeed $L(2, 1)$-labelings. The next result improves the upper bound of Theorem 2.4 for the λ-numbers of generalized flowers with $n = 2$.

$q \geq 1$		$P_2 \square C_{3q}$	$P_2 \square C_{1+3q}$	$P_2 \square C_{2+3q}$
$p = 2$:	petal 1	$\begin{matrix}0&2&5\\4&6&1\end{matrix}$	$\begin{matrix}0&2&7&5\\4&6&3&1\end{matrix}$	$\begin{matrix}0&2&4&7&5\\4&6&0&3&1\end{matrix}$
	petal 2	$\begin{matrix}0&3&6\\4&7&2\end{matrix}$	$\begin{matrix}0&3&1&6\\4&7&5&2\end{matrix}$	$\begin{matrix}0&3&1&4&6\\4&7&5&0&2\end{matrix}$
$p = 3$:	petal 1	$\begin{matrix}0&2&5\\8&4&1\end{matrix}$	$\begin{matrix}0&2&8&5\\8&4&6&1\end{matrix}$	$\begin{matrix}0&2&7&3&5\\8&4&0&6&1\end{matrix}$
	petal 2	$\begin{matrix}0&3&6\\8&5&2\end{matrix}$	$\begin{matrix}0&3&1&6\\8&5&7&2\end{matrix}$	$\begin{matrix}0&3&1&4&6\\8&5&7&0&2\end{matrix}$
	petal 3	$\begin{matrix}0&4&7\\8&6&3\end{matrix}$	$\begin{matrix}0&4&2&7\\8&6&0&3\end{matrix}$	$\begin{matrix}0&4&8&1&7\\8&6&0&5&3\end{matrix}$
$p \geq 4$:	petal 1	$\begin{matrix}0&p+1&2p+1\\2p+2&2p&p\end{matrix}$	$\begin{matrix}0&p+1&p+3&2p+1\\2p+2&2p&0&p\end{matrix}$	$\begin{matrix}0&p+1&p+3&1&2p+1\\2p+2&2p&0&p+2&p\end{matrix}$
	petal k $k = 2, 3, \ldots, p$	$\begin{matrix}0&k&p+k\\2p+2&p+k-1&k-1\end{matrix}$	$\begin{matrix}0&k&p+k+2&p+k\\2p+2&p+k-1&k+1&k-1\end{matrix}$	$\begin{matrix}0&k&k+2&p+k+2&p+k\\2p+2&p+k-1&0&k+1&k-1\end{matrix}$

Fig. 4.1. $L(2, 1)$-labelings of $F_2(m_1, m_2, \ldots, m_p)$ (the shaded block with the last, 0th, and 1st columns in each matrix is repeated $q \geq 1$ times).

$p = 2$: petal 1	0	2	5	0	2	4	7	5
	4	6	1	4	6	0	3	1

Fig. 4.2. 7-labeling of $P_2 \square C_8$.

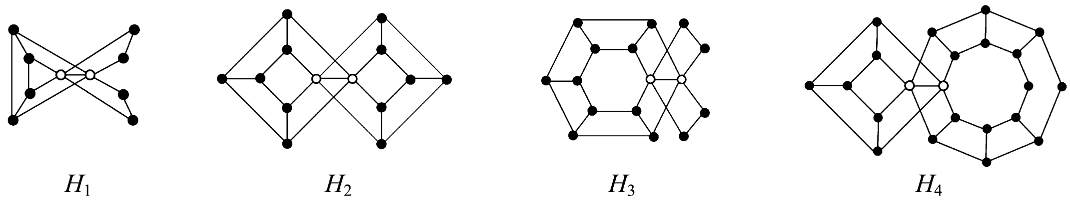


Fig. 4.3. Useful graphs with λ-number 7.

Lemma 4.1. *Let $p \geq 2$ and $3 \leq m_1 \leq m_2 \leq \cdots \leq m_p$ be integers and let $G = F_2(m_1, m_2, \ldots, m_p)$. Then $\lambda(G) \leq 7$ if $p = 2$, otherwise $\lambda(G) \leq 2p + 2$.*

Proof. The result follows from the $L(2, 1)$-labelings of G given in Fig. 4.1 in which each array of labels with 2 rows and $i = 3$, 4, or 5 columns is used for a petal isomorphic to $P_2 \square C_m$ (0th column used for the stem) where the shaded block consisting of the last, 0th, 1st columns can be replaced with $q \geq 1$ copies of itself arranged consecutively as needed to reach the desired value of m. For example, if $p = 2$ and $m = 8$, then $m = 2 + 3q$ for $q = 2$ so the corresponding matrix in Fig. 4.1 for petal 1 produces the 7-labeling in Fig. 4.2. □

We will first focus on the case $p = 2$ as it offers more complexity.

Lemma 4.2. *Let $3 \leq m_1 \leq m_2$ be integers and let $G = F_2(m_1, m_2)$. If $\{m_1, m_2\} \cap \{3, 6\} \neq \varnothing$ or $(m_1, m_2) \in \{(4, 4), (4, 8)\}$, then $\lambda(G) = 7$; otherwise $\lambda(G) = 6$.*

Proof. We used a computer program to verify that all the graphs in Fig. 4.3 have λ-number 7. (Note: Each of these graphs is minimal in the sense that any proper subgraph will have λ-number 6 or less.) If $\{m_1, m_2\} \cap \{3, 6\} \neq \varnothing$ or $(m_1, m_2) \in \{(4, 4), (4, 8)\}$, then G contains H_i as a subgraph for some $i = 1, 2, 3, 4$ (white vertices in the stem). Hence $\lambda(G) \geq 7$ and the equality follows from Lemma 4.1.

Suppose $\{m_1, m_2\} \cap \{3, 6\} = \varnothing$ and $(m_1, m_2) \notin \{(4, 4), (4, 8)\}$. Since $\lambda(G) \geq \Delta(G) + 1 = 6$, it is sufficient to exhibit a 6-labeling of G to conclude that $\lambda(G) = 6$. If $(m_1, m_2) = (8, 8)$, then Fig. 4.4 contains a 6-labeling of G. If $(m_1, m_2) \neq (8, 8)$, then it is possible to choose one 6-labeling of $P_2 \square C_{m_1}$ from one of the columns of the table in Appendix A, and another 6-labeling of $P_2 \square C_{m_2}$ from the other column to obtain a 6-labeling of G. □

Theorem 4.3. *Let $p \geq 2$ and $3 \leq m_1 \leq m_2 \leq \cdots \leq m_p$ be integers and let $G = F_2(m_1, m_2, \ldots, m_p)$. If $p = 2$ and $[\{m_1, m_2\} \cap \{3, 6\} \neq \varnothing$ or $(m_1, m_2) \in \{(4, 4), (4, 8)\}]$, then $\lambda(G) = 7$; otherwise $\lambda(G) = 2p + 2$.*

F_2(8,8): petal 1

0	2	6	1	4	2	6	3
6	4	0	3	6	0	4	1

petal 2

0	4	6	3	0	6	2	5
6	2	0	5	2	4	0	3

Fig. 4.4. 6-labeling of $F_2(8, 8)$.

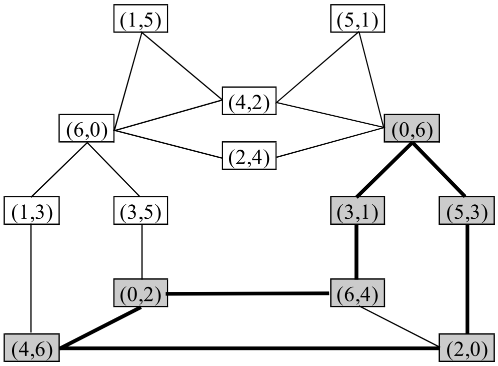


Fig. 4.5. Subgraph of G^* used to generate the table in Appendix A
(shaded vertices and thick edges correspond to the 6-labeling in the second column of the table for $m = 7$).

Proof. The case $p = 2$ is proved in Lemma 4.2. If $p > 2$, then $\lambda(G) \geq \Delta(G) + 1 = 2p + 2$. But Lemma 4.1 implies $\lambda(G) \leq 2p + 2$, and therefore the desired equality follows. □

We will close this section offering an overview on how we obtained the table in Appendix A as the computer program alone was unable to derive an equivalent set of 6-labelings due to the large number of possibilities. Our goal was to construct 6-labelings of $G = F_2(m_1, m_2)$ for certain values of m_1 and m_2 that had reasonably compact descriptions and were preferably extensions of the same partial labeling of the subgraph of G induced by the vertices on the stem and their neighbors. The two vertices on the stem have to be labeled with 0 and 6 as they are the only vertices of G achieving the maximum degree 5. The next step was to find labels to assign to the four neighbors of the vertices in the stem within a petal such that this partial labeling could be extended to the entire petal, regardless of the size of the petal. Note that the four vertices outside the stem and adjacent to the vertex on the stem labeled 0 (resp., 6) must be labeled 2, 3, 4, 5 (resp., 1, 2, 3, 4). Consider the auxiliary graph G^* with vertices (i, j) where $i, j \in \{0, 1, \ldots, 6\}$ are at least two apart, and edges connect pairs of vertices (i, j) and (i', j') so that i and i' are at least two apart, j and j' are at least two apart, $i \neq j'$, and $i' \neq j$. Therefore, if (i, j) and (i', j') are adjacent in G^*, then the transpose of these two pairs could be used as two consecutive columns in an $L(2, 1)$-labeling of a petal of G. We would like to find closed paths (not necessarily simple) of order m_ℓ in G^* for $\ell = 1, 2$ both containing a vertex $(0, 6)$ so that the vertices immediately before and after this vertex on each closed path have all different coordinates; hence the two $L(2, 1)$-labelings corresponding to these closed paths can be assigned to each petal of G, respectively, generating the desired 6-labeling. By inspection, G^* contains the subgraph in Fig. 4.5 and each 6-labeling in the table in Appendix A was obtained from a closed path in this subgraph. For example, the closed path consisting of the shaded vertices and thick edges corresponds to the 6-labeling on the second column of the table in Appendix A for $m = 7$.

5. The λ-number of generalized flowers with $n = 3$ and 4

We will first focus on the case $n = 4$ since most of the case $n = 3$ will follow directly from the former case. Even though each result has a structure analogous to the structure of its counterpart in the case $n = 2$, their proofs are in general more involved. The first result improves the upper bound of Theorem 2.4 for the λ-numbers of generalized flowers with $n = 4$.

Lemma 5.1. *Let $p \geq 2$ and $3 \leq m_1 \leq m_2 \leq \cdots \leq m_p$ be integers and let $G = F_4(m_1, m_2, \ldots, m_p)$. Then $\lambda(G) \leq 8$, if $p = 2$, otherwise $\lambda(G) \leq 2p + 3$.*

Proof. If $p = 2$, Theorem 2.4 implies $\lambda(G) \leq 2p + 4 = 8$. If $p > 2$, the result follows from the $(2p + 3)$-labelings of G given in Fig. 5.1 in which each array of labels with 4 rows and $i = 3, 4$, or 5 columns is used for a petal isomorphic to $P_4 \,\square\, C_m$ (0th column used for the stem) where the block consisting of the last, 0th, 1st columns can be replaced with $q \geq 1$ copies of itself arranged consecutively as needed to reach the desired value of m. □

The next result focuses on the case $p = 2$.

Lemma 5.2. *Let $3 \leq m_1 \leq m_2$ be integers and let $G = F_4(m_1, m_2)$. If $(m_1, m_2) \in \{(3, 3), (3, 5), (3, 6), (4, 4), (4, 5)\}$, then $\lambda(G) = 8$; otherwise $\lambda(G) = 7$.*

$q \geq 1$		$P_4 \square C_{3q}$			$P_4 \square C_{1+3q}$				$P_4 \square C_{2+3q}$				
$p = 3$:	petal 1	5	7	1	5	7	9	1	5	7	0	3	1
		0	2	6	0	2	4	6	0	2	4	8	6
		9	5	3	9	5	8	3	9	5	7	0	3
		4	0	7	4	0	2	7	4	0	2	5	7
	petal 2	5	8	2	5	8	0	2	5	8	4	0	2
		0	3	7	0	3	9	7	0	3	1	5	7
		9	6	1	9	6	4	1	9	6	8	3	1
		4	2	8	4	2	0	8	4	2	0	6	8
	petal 3	5	9	3	5	9	1	3	5	9	0	7	3
		0	4	8	0	4	6	8	0	4	6	1	8
		9	7	2	9	7	0	2	9	7	3	5	2
		4	1	6	4	1	3	6	4	1	8	0	6
$p \geq 4$:	petal 1	$2p+2$	$2p$	p	$2p+2$	$2p$	0	p	$2p+2$	$2p$	0	$p+2$	p
		0	$p+1$	$2p+1$	0	$p+1$	$p+3$	$2p+1$	0	$p+1$	$p+3$	1	$2p+1$
		$2p+3$	2	$p+2$	$2p+3$	2	4	$p+2$	$2p+3$	2	4	$2p$	$p+2$
		1	$p+3$	3	1	$p+3$	0	3	1	$p+3$	0	5	3
	petal k $k = 2, 3, \ldots, p$	$2p+2$	$p+k-1$	$k-1$	$2p+2$	$p+k-1$	$k+1$	$k-1$	$2p+2$	$p+k-1$	0	$k+1$	$k-1$
		0	k	$p+k$	0	k	$p+k+2$	$p+k$	0	k	$k+2$	$p+k+2$	$p+k$
		$2p+3$	$p+k+1$	$k+1$	$2p+3$	$p+k+1$	$k+3$	$k+1$	$2p+3$	$p+k+1$	$\boxed{\mathbf{p+k+3}}$	$k+3$	$k+1$
		1	$k+2$	$p+k+2$	1	$k+2$	0	$p+k+2$	1	$k+2$	k	0	$p+k+2$
									If $k = p$, set bolded label to 1				

Fig. 5.1. $(2p + 3)$-labelings of $F_4(m_1, m_2, \ldots, m_p)$ with $p > 2$ (the block with the last, 0th, and 1st columns in each matrix is repeated $q \geq 1$ times).

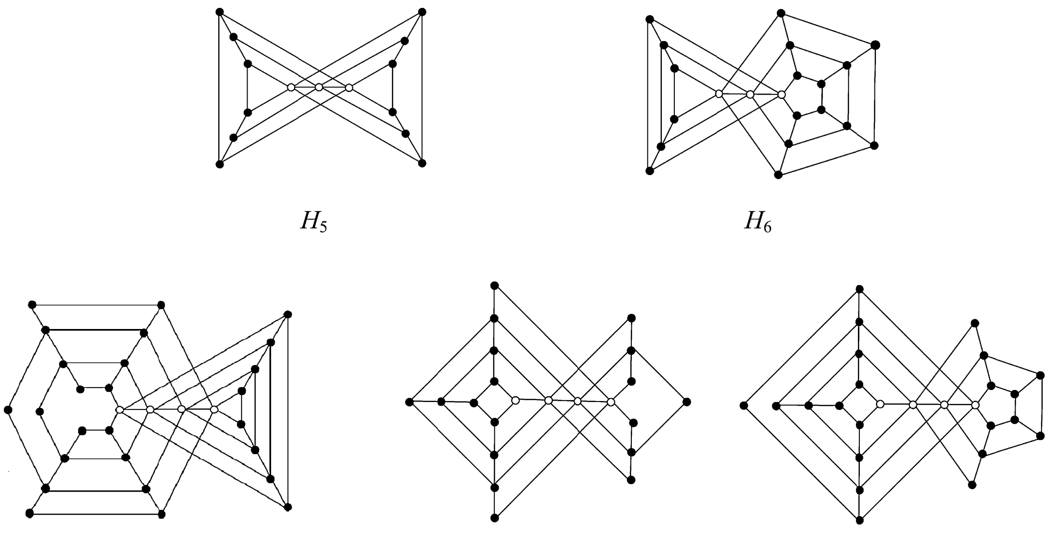


Fig. 5.2. Useful graphs with λ-number 8.

Proof. We used a computer program to verify that all the graphs in Fig. 5.2 have λ-number 8. (Note: Each of these graphs is minimal in the sense that any proper subgraph will have λ-number 7 or less.)

If $(m_1, m_2) \in \{(3, 3), (3, 5), (3, 6), (4, 4), (4, 5)\}$, then G contains an H_i for some $i = 5, 6, \ldots, 9$ as a subgraph (white vertices in the stem). Hence $\lambda(G) \geq 8$ and the equality follows from Lemma 5.1.

Suppose $(m_1, m_2) \notin \{(3, 3), (3, 5), (3, 6), (4, 4), (4, 5)\}$. Since $\lambda(G) \geq \Delta(G)+1 = 7$, it is sufficient to exhibit a 7-labeling of G to conclude that $\lambda(G) = 7$. If $(m_1, m_2) = (5, 7)$, then Fig. 5.3 contains a 7-labeling of G.

If $(m_1, m_2) = (6, 6)$, then Fig. 5.4 contains a 7-labeling of G.

If $m_1 = 5$ and $m_2 \neq 7$, then label petal 1 with the 7-labeling in Fig. 5.5 and label petal 2 with the 7-labeling of $P_4 \square C_{m_2}$ from the third column of the table in Appendix B to obtain a 7-labeling of G.

F_4(5,7): petal 1

4	1	5	0	2
0	6	3	7	5
7	2	0	4	1
3	5	7	2	6

petal 2

4	6	3	7	2	0	7
0	2	5	1	4	6	3
7	4	0	3	7	1	5
3	1	7	5	2	4	0

Fig. 5.3. 7-labeling of $F_4(5, 7)$.

F_4(6,6): petal 1

4	6	2	0	5	1
0	3	7	4	2	6
7	5	1	6	0	3
2	0	4	2	7	5

petal 2

4	7	1	4	0	2
0	2	5	7	3	5
7	4	0	2	6	1
2	6	3	5	0	4

Fig. 5.4. 7-labeling of $F_4(6, 6)$.

$F_4(5, m_2)$: petal 1

4	6	0	3	7
0	3	7	5	2
7	5	2	0	4
3	0	4	6	1

Fig. 5.5. 7-labeling for petal 1 of $F_4(5, m_2)$.

F_3(3,6): petal 1

0	2	5
7	4	1
3	0	6

petal 2

0	3	7	5	1	4
7	5	0	3	6	2
3	1	4	7	0	5

F_3(4,4): petal 1

1	3	7	5
7	5	0	2
4	1	3	6

petal 2

1	4	2	6
7	0	5	3
4	2	7	0

F_3(4,5): petal 1

1	3	6	4
7	5	0	2
4	1	3	6

petal 2

1	5	7	2	6
7	3	1	4	0
4	0	5	7	2

Fig. 5.6. 7-labelings of $F_3(3, 6)$, $F_3(4, 4)$, and $F_3(4, 5)$.

Finally, if $m_1 \neq 5$ and $(m_1, m_2) \neq (6, 6)$, then it is possible to choose one 7-labeling of $P_4 \,\square\, C_{m_1}$ from one of the first two columns of the table in Appendix B, and another 7-labeling of $P_4 \,\square\, C_{m_2}$ from the other column (also from the first two) to obtain a 7-labeling of G. $\square$

Theorem 5.3. *Let $p \geq 2$ and $3 \leq m_1 \leq m_2 \leq \cdots \leq m_p$ be integers and let $G = F_4(m_1, m_2, \ldots, m_p)$. If $p = 2$ and $(m_1, m_2) \in \{(3, 3), (3, 5), (3, 6), (4, 4), (4, 5)\}$, then $\lambda(G) = 8$; otherwise $\lambda(G) = 2p + 3$.*

Proof. The case $p = 2$ is proved in Lemma 5.2. If $p > 2$, then $\lambda(G) \geq \Delta(G) + 1 = 2p + 3$ and the equality follows from Lemma 5.1. $\square$

Theorem 5.4. *Let $p \geq 2$ and $3 \leq m_1 \leq m_2 \leq \cdots \leq m_p$ be integers and let $G = F_3(m_1, m_2, \ldots, m_p)$. If $p = 2$ and $(m_1, m_2) \in \{(3, 3), (3, 5)\}$, then $\lambda(G) = 8$; otherwise $\lambda(G) = 2p + 3$.*

Proof. If $p = 2$ and $(m_1, m_2) \in \{(3, 3), (3, 5)\}$, then G contains H_5 or H_6 as a subgraph, hence $\lambda(G) \geq 8$ and the equality follows from Theorem 2.4.

Suppose $p \neq 2$ or ($p = 2$ and $(m_1, m_2) \notin \{(3, 3), (3, 5)\}$). Since G is a subgraph of $F_4(m_1, m_2, \ldots, m_p)$, then Theorem 5.3 implies that $\lambda(G) \leq 2p + 3$ except when $p = 2$ and $(m_1, m_2) \in \{(3, 6), (4, 4), (4, 5)\}$, therefore equality holds since $\lambda(G) \geq \Delta(G) + 1 = 2p + 3$. Combining this last general lower bound and the 7-labelings of G in Fig. 5.6 for $p = 2$ and $(m_1, m_2) \in \{(3, 6), (4, 4), (4, 5)\}$, we also obtain $\lambda(G) = 2p + 3$. $\square$

6. Concluding remarks

We completely characterized the λ-number of generalized flowers in Theorem 1.1 which summarizes the results in Corollary 2.6 and Theorems 4.3, 5.3, and 5.4. To determine a tight general upper bound for this number, we introduced the notion of extending a circular $L(d + 1, 1)$-labeling of a graph G to an $L(d, 1)$-labeling of $P_n \,\square\, G$ without increasing the span of labels used. This approach unifies a series of seemingly disparate techniques found in the literature to determine upper bounds for the λ-number of $P_n \square G$ for different families of graphs G. Similar approaches may be useful in investigating the λ-number of the Cartesian product of other families of graphs.

In closing, we would like to remark that the λ-numbers of amalgamations of rectangular grids along a path determined in [1] could be derived from the results in this manuscript as these amalgamations are subgraphs of the amalgamations of cylindrical rectangular grids along a path studied here.

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Appendix A

6-labelings of $P_2 \square C_m$ for the two petals of $F_2(m_1, m_2)$ when $\{m_1, m_2\} \cap \{3, 6\} = \varnothing$ and $(m_1, m_2) \notin \{(4, 4), (4, 8), (8, 8)\}$ used in Lemma 4.2 (the two 6-labelings must belong to different columns).

Cycle order	**Extension of** **0 2 … 4** **6 4 … 2**	**Extension of** **0 3 … 5** **6 1 … 3**
$m = 3$		
$m = 4$	**0 2** 6 **4** **6 4** 0 **2**	
$m = 5$	**0 2** 6 1 **4** **6 4** 0 5 **2**	**0 3** 6 2 **5** **6 1** 4 0 **3**
$m = 6$		
$m = 7$	**0 2** 5 0 2 6 **4** **6 4** 1 3 5 0 **2**	**0 3** 6 0 4 2 **5** **6 1** 4 2 6 0 **3**
$m = 8$	**0 2** 6 4 0 2 6 **4** **6 4** 0 2 6 4 0 **2** Note: two extensions of order 4	
$m = 9$	**0 2** 6 4 0 2 6 1 **4** **6 4** 0 2 6 4 0 5 **2** Note: extensions of order 4 and 5	**0 3** 6 2 4 0 6 2 **5** **6 1** 4 0 6 2 4 0 **3**
$m = 10$	**0 2** 6 1 4 0 2 6 1 **4** **6 4** 0 5 2 6 4 0 5 **2** Note: two extensions of order 5	**0 3** 6 2 5 0 3 6 2 **5** **6 1** 4 0 3 6 1 4 0 **3** Note: two extensions of order 5
$m = 11$	For $m \geq 11$: Extension of order 4 and extension of order $m - 4$	**0 3** 6 0 4 2 6 0 4 2 **5** **6 1** 4 2 6 0 4 2 6 0 **3**
$m = 12$		**0 3** 6 2 5 0 3 6 0 4 2 **5** **6 1** 4 0 3 6 1 4 2 6 0 **3** Note: extensions of order 5 and 7
$m = 13$		**0 3** 6 2 4 0 5 2 0 3 6 2 **5** **6 1** 4 0 6 3 1 4 6 1 4 0 **3**
$m = 14$		For $m \geq 14$: Extension of order 5 and extension of order $m - 5$

Appendix B

7-labelings of $P_4 \square C_m$ for the petals of $F_4(m_1, m_2)$ when $(m_1, m_2) \notin \{(3,3), (3,5), (3,6), (4,4), (4,5), (5,7), (6,6)\}$ used in Lemma 5.2.

<table>
<tr><td>Cycle order</td><td>Extension of
4 2 … 7
0 6 … 3
7 4 … 1
3 0 … 5</td><td>Extension of
4 6 … 1
0 2 … 5
7 5 … 2
3 1 … 6</td><td>Extension of
4 1 … 2
0 6 … 5
7 2 … 1
3 5 … 6</td></tr>
<tr><td>m = 3</td><td>4 2 7
0 6 3
7 4 1
3 0 5</td><td></td><td></td></tr>
<tr><td>m = 4</td><td></td><td>4 6 3 1
0 2 7 5
7 5 0 2
3 1 4 6</td><td></td></tr>
<tr><td>m = 5</td><td></td><td></td><td>4 1 5 0 2
0 6 3 7 5
7 2 0 4 1
3 5 7 2 6</td></tr>
<tr><td>m = 6</td><td>4 2 7 4 2 7
0 6 3 0 6 3
7 4 1 7 4 1
3 0 5 3 0 5
Note: two extensions of order 3</td><td></td><td>4 1 7 4 0 2
0 6 3 1 7 5
7 2 0 5 3 1
3 5 7 2 0 6</td></tr>
<tr><td>m = 7</td><td>4 2 5 1 4 2 7
0 6 3 7 0 6 3
7 4 1 5 2 4 1
3 0 7 3 6 0 5</td><td>4 6 3 5 0 3 1
0 2 7 1 4 7 5
7 5 0 3 6 0 2
3 1 4 7 1 4 6</td><td></td></tr>
<tr><td>m = 8</td><td>4 2 5 0 6 3 0 7
0 6 3 7 4 1 6 3
7 4 1 5 2 7 4 1
3 0 7 3 6 0 2 5</td><td>4 6 3 1 4 6 3 1
0 2 7 5 0 2 7 5
7 5 0 2 7 5 0 2
3 1 4 6 3 1 4 6
Note: two extensions of order 4</td><td>4 1 5 0 4 6 0 2
0 6 3 7 1 3 7 5
7 2 0 4 6 0 4 1
3 5 7 1 3 5 2 6</td></tr>
<tr><td>m = 9</td><td>For m ≥ 9:
Extension of order 3 and extension of order m − 3</td><td>4 6 0 2 6 4 2 6 1
0 2 7 5 3 0 7 3 5
7 5 3 0 7 2 5 0 2
3 1 6 2 4 6 1 4 6</td><td>4 1 7 3 0 6 3 0 2
0 6 4 1 7 4 1 7 5
7 2 0 6 3 0 5 3 1
3 5 7 2 5 7 2 0 6</td></tr>
<tr><td>m = 10</td><td></td><td>4 6 3 0 5 2 0 6 3 1
0 2 7 4 1 6 4 2 7 5
7 5 0 6 3 0 7 5 0 2
3 1 4 2 7 5 3 1 4 6</td><td>4 1 5 0 2 4 1 5 0 2
0 6 3 7 5 0 6 3 7 5
7 2 0 4 1 7 2 0 4 1
3 5 7 2 6 3 5 7 2 6
Note: two extensions of order 5</td></tr>
<tr><td>m = 11</td><td></td><td>For m ≥ 11:
Extension of order 4 and extension of order m − 4</td><td>4 1 5 0 2 4 1 7 4 0 2
0 6 3 7 5 0 6 3 1 7 5
7 2 0 4 1 7 2 0 5 3 1
3 5 7 2 6 3 5 7 2 0 6
Note: extensions of order 5 and 6</td></tr>
<tr><td>m = 12</td><td></td><td></td><td>4 1 7 4 0 2 4 1 7 4 0 2
0 6 3 1 7 5 0 6 3 1 7 5
7 2 0 5 3 1 7 2 0 5 3 1
3 5 7 2 0 6 3 5 7 2 0 6
Note: two extensions of order 6</td></tr>
<tr><td>m = 13</td><td></td><td></td><td>For m ≥ 13:
Extension of order 5 and extension of order m − 5</td></tr>
</table>

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