Climate & Chaos 2022 **(Due Wednesday)**

Climate & Chaos Homework 1

Instructor: Jessie

Write up at least one problem!

(1) Nighttime Temperature

Using Model 2 from class, estimate how much the atmosphere in the model would cool overnight. The heat capacity of the atmosphere is about 10^7 J/(K$\cdot$ m^2).

(Hint: It's not much, so you can make some assumptions that make things easier! What radiation, if any, is coming into the top of the atmosphere? What radiation, if any, is going out?)

(2) Budyko's Model

Satellite data suggests that blackbody radiation might not be the best model for Earth's outgoing radiation! Instead, we can use a linear(-ish) model: $A + BT_s$. (The ish is because A and B *also* depend on temperature if we're being careful, as we'll consider in (c) and (d).)

(a) Modify Model 1 to use this form for Earth's outgoing radiation instead, and find an expression for surface temperature in this case. We'll call this temperature T_0.

(b) Suppose that S_0 changes slightly by an amount we'll call ΔS. If α, A, and B don't change, write an expression for ΔT, how much T_s changes.

(c) Suppose now that α, A, and B *do* change with temperature as follows:

$$\alpha(T) = \alpha_0 + \alpha_1 \Delta T;$$
$$A(T) = A_0 + A_1 \Delta T;$$
$$B(T) = B_0 + B_1 \Delta T;$$

where α_0, A_0, B_0 are the values when $T = T_0$. Find an expression relating ΔS to ΔT. You can assume that $(\Delta T)^2 \approx 0$.

(d) With some wrangling of terms, you can get this to look like

$$B_0 \Delta T = f(1 - \alpha_0)\Delta S,$$

with f condensing a pretty messy expression. Compare this to your answer to (b). Why do you think f is called "climate gain?"

(3) Difference Equations and Recurrence Relations

Suppose that you have a sequence of values $a_0, a_1, \cdots a_n, \cdots$. In general, let $\Delta a_n = a_{n+1} - a_n$.

(a) Rewrite $2\Delta a_n + 4a_n = 0$ as a recurrence relation in terms of a_{n+1} and a_n, and find the first few terms of a sequence that satisfies this.

(b) The second difference, Δ^2, takes the difference of consecutive first differences. (In other words, imagine a new sequence formed by first differences of the original sequence, and now we take differences of that.) How would you write $\Delta^2 a_n$ in terms of a_{n+2}, a_{n+1}, and a_n?

(c) Rewrite $3\Delta^2 a_n + 4\Delta a_n + 7a_n = 0$ as a recurrence relation.

Climate & Chaos 2022 **(Due Thursday)**

Climate & Chaos Homework 2

Instructor: Jessie

Write up at least one problem!

(1) Fixed Points and Stability

Suppose you have the system $x_{n+1} = ax_n + b$, where a, b are real numbers.

(a) When does this system have a fixed point? If one exists, describe the value x^* of that point in terms of a and b.

(b) Can this system have multiple fixed points? Why or why not?

(c) How do a and/or b relate to the stability of the fixed point? Make a conjecture and try to prove it!

(2) More Fixed Points and Stability

Now consider the systems $x_{n+1} = x_n^2$ and $\Delta y_{n+1} = y_n^2$.

(a) Find the fixed point(s) of the first system.

(b) For any fixed points you found in (a), what is their stability, and why do you think so?

(c) Find any fixed point(s) of the second system.

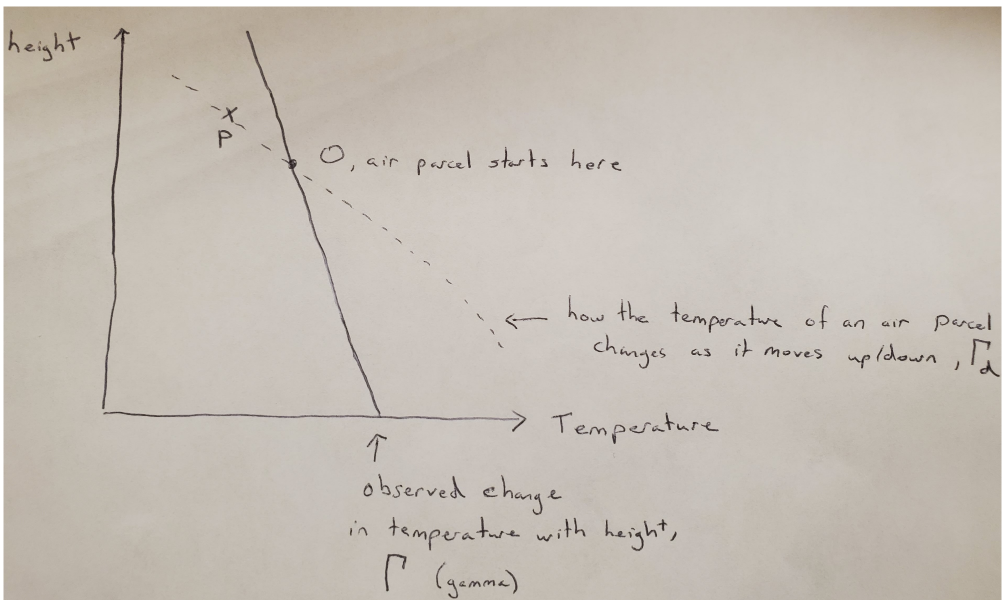
(d) For any fixed points you found in (c), what is their stability, and why do you think so?

(3) Atmospheric Stability

You may have heard someone say that hot air rises. This is true but not as simple as it sounds! An air parcel changes temperature as it moves up or down, and so does the surrounding air, but not at the same rate. An air parcel tends to "settle" where its temperature matches the surrounding temperature, so that's where it's at equilibrium.

(a) In the diagram on the other side of the page, an air parcel starts at point O, an equilibrium point where the air parcel's temperature matches the surrounding air temperature. Is point O a stable equilibrium, and why or why not? (Think about what happens if the parcel gets bumped up to point P.)

(b) In the diagram, Γ was steeper than Γ_d. Draw a diagram where the opposite is true. Is the equilibrium point stable in that case? Why or why not?


height
x
P
O, air parcel starts here
how the temperature of an air parcel changes as it moves up/down, Γ_d
Temperature
observed change in temperature with height, Γ (gamma)

Climate & Chaos 2022 **(Due Friday)**

Climate & Chaos Homework 3

Instructor: Jessie

Write up at least one problem!

(1) Look at these maps! You can draw cobweb diagrams and look for things like fixed points, periodic orbits, stability, and any other interesting behavior.

(a) $x_{n+1} = \frac{2x_n}{1+x_n}$

(b) $x_{n+1} = \sqrt{x_n}$

(c) $x_{n+1} = 3x_n - x_n^3$

(2) Decimal Shift Map

Let $x \mod 1 = x - \lfloor x \rfloor$, or the decimal part of x. Then consider the map $x_{n+1} = 10x_n \mod 1$.

(a) Find the fixed points of the map.

(b) Show that there is a periodic point of period p for all integers $p > 1$.

(c) Are these fixed points and periodic orbits stable? Why or why not?

(d) Explain why the map has infinitely many points that aren't periodic.

Climate & Chaos 2022 **(Due Saturday)**

Climate & Chaos Homework 4

Instructor: Jessie

Write up at least one problem!

(1) Logistic Map and Quadratic Map: Interesting Points

The **logistic map** $x_{n+1} = rx_n(1-x_n)$ is sometimes used as a model of population growth, where x_n is between 0 and 1 and represents the population size relative to the maximum possible population. The parameter r is typically given a value between 0 and 4.

The **quadratic map** is $y_{n+1} = y_n^2 + c$, where c is a parameter that can be any real number.

(a) Find the fixed points of the logistic map.

(b) For $r < 1$, what is the stability of those fixed points? What about for $1 \leq r \leq 3$?

(c) Find the fixed points of the quadratic map.

(d) For $c = -1$, find a point of period 2 in the quadratic map.

(2) Logistic Map and Quadratic Map: Transformation

(Read the intro to the maps above if you didn't already!)

(a) These maps behave the same way for particular values of r and c! We can prove this by transforming one into the other. Let $x_n = ay_n + b$, for some constants a and b that you'll determine. Recover the form of the logistic map from the form of the quadratic map.

(b) The logistic map is chaotic for most values of r in $[3.56995, 4]$. Using your result in (a), what is c if $r = 4$?

(3) Period 3 without Chaos

We've been looking at maps from the real line to itself. For this problem, think about the boundary of a circle instead. (We'll call that boundary just "the circle" from here.)

(a) Describe at least three very different functions that map points on a circle to points on a circle.

(b) Here, we can have points of period 3 without chaos! Find a map on the circle with at least one set of period 3 points.