Constrained Optimization

Students worked on these problems in small groups (at tables/desks or at boards) in a recitation section for Multivariable Calculus for engineering students (mostly first- and second-year undergraduates). My recitations ranged in size from 15 to 30 students. I circulated around the room while students worked, answering questions when students had them, listening in on discussions, and asking students about their thought processes. Small-group work was followed by a full-class discussion of the main ideas.

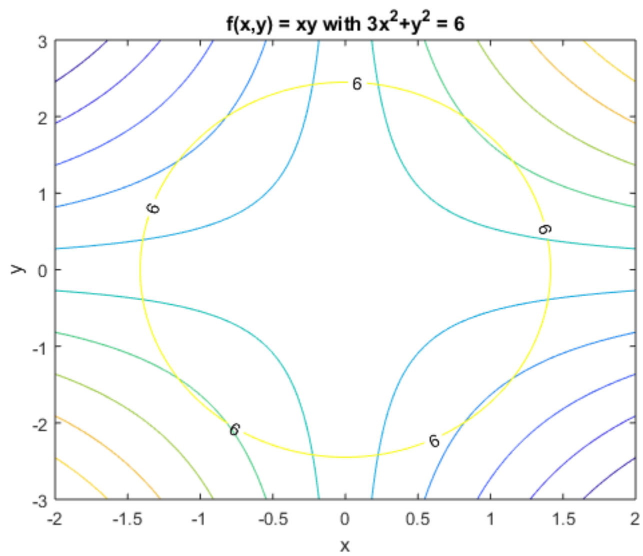
The students were introduced to constrained optimization using Lagrange multipliers in lecture. The main goals of this set of problems were to connect the algebraic setup of Lagrange multipliers to a graphical representation and to scaffold the process of using Lagrange multipliers for optimization.

While not asked for in the problem, a lot of students think about how to determine which points are maxima and which are minima. This makes a good question for the full-class discussion.

Problem 1

This problem focuses on connecting graphical representations with the ideas being optimization with Lagrange multipliers. I introduced the lesson by talking about Lagrange multipliers with the idea of hiking a certain loop around a mountain and physically demonstrated the idea, using my arms to show the gradients of the two functions. (In reviewing for exams, students frequently referred to this demonstration.) The idea of parallel gradients should then be fresh in students' minds, but they have to apply it to a different-looking context.

The figure below shows the curve $3x^2 + y^2 = 6$ and contour lines of $f(x, y) = xy$. Mark in the figure where $f(x, y)$ is locally maximal and minimal on $3x^2 + y^2 = 6$. Explain how you know.



Problem 2

This is a more procedural/mechanical problem. The first problem focuses on what it looks like and means for the gradients to be parallel, and now the students use that to calculate extrema.

Only the first four parts of this question are necessary for students to work through the main ideas; the following full-class discussion zooms out from this particular problem to talk about the steps students took. Parts 5-7 are an extension. Students do solve problems with inequality constraints in the course, so they see the idea of checking the interior and the boundary separately for extrema. Parts 5-7 are a variation on that (with a sense of "interior" that is less familiar to students) and so provide a challenge for groups that are comfortable with the mechanics of Lagrange multipliers already.

Consider the function $f(x, y) = x^2 - xy + y^2$ with the constraint $x^2 + y^2 = 1$. We are interested in finding the maximum and minimum values of this function.

1. Find the system of equations for the extreme values of f subject to the constraint.
2. Solve the system for x and y. What are all the points where maximum or minimum values occur?
3. What is the maximum? Where does it occur?
4. What is the minimum? Where does it occur?
5. Now suppose we had an additional constraint of $x, y \geq 0$. Which of the points you found still works?
6. The additional constraint introduces boundary points that might give new extreme values of f under the constraints. What are the boundary points?
7. What is f at those boundary points? Are they extreme values of f under the constraints, and if so, what kind?