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Labeling amalgamations of Cartesian products of complete graphs with a condition at distance two

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ABSTRACT

The spectrum allocation problem in wireless communications can be modeled through vertex labelings of a graph, wherein each vertex represents a transmitter and edges connect vertices whose corresponding transmitters are operating in close proximity. One well-known model is the $L(2, 1)$-labeling of a graph G in which a function f maps the vertices of G to the nonnegative integers such that if vertices x and y are adjacent, then $|f(x) - f(y)| \geq 2$, and if x and y are at distance two, then $|f(x) - f(y)| \geq 1$. The λ-number of G is the minimum span over all $L(2, 1)$-labelings of G. Informally, an amalgamation of two graphs G_1 and G_2 along a fixed graph G_0 is the simple graph obtained by identifying the vertices of two induced subgraphs isomorphic to G_0, one in G_1 and the other in G_2. In this work, we supply a tight upper bound for the λ-number of amalgamations of several Cartesian products of complete graphs along a complete graph and find the exact λ-numbers for certain infinite subclasses of amalgamations of this form. A surprising relationship between the former upper bound and the minimum makespan scheduling problem is highlighted.

1. Introduction

An $L(2, 1)$-*labeling* of a graph G is an assignment of non-negative integers to its vertices such that adjacent vertices must receive integers at least two apart, and vertices at distance two must receive integers at least one apart. The study of $L(2, 1)$-labelings and their variations was motivated by the channel assignment problem [10] and has generated a vast literature since these labelings were introduced in 1992 [9]. We refer the reader to the surveys in [2,17] and a sample of the most recent works in the field in [3,4,12–16].

A k-*labeling* of a graph G is an $L(2, 1)$-labeling that uses labels in the set $\{0, 1, \ldots, k\}$. The minimum k so that G has a k-labeling is called the λ-*number* of G and will be denoted by $\lambda(G)$. The long-standing conjecture in the field states that $\lambda(G) \leq \Delta^2(G)$, where $\Delta(G)$ denotes the maximum degree of G [9]. This conjecture holds for graphs with $\Delta(G)$ larger than approximately 10^{69} [11] and for graphs with at most $(\lfloor \Delta(G)/2 \rfloor + 1)(\Delta^2(G) - \Delta(G) + 1) - 1$ vertices [4]. The best known general upper bound is $\lambda(G) \leq \Delta^2(G) + \Delta(G) - 2$ [8]. Even though the general problem of determining $\lambda(G)$ is NP-hard [7], several bounds and exact λ-numbers for different families of graphs are known. One of these families is the class of amalgamations of graphs studied in [1].

Definition 1.1. Let $G_1, G_2, \ldots, G_p$ be $p \geq 2$ pairwise disjoint graphs each containing a fixed induced subgraph isomorphic to a graph G_0. The *amalgamation* of $G_1, G_2, \ldots, G_p$ *along* G_0 is the simple graph $G = Amalg(G_0; G_1, G_2, \ldots, G_p)$ obtained by

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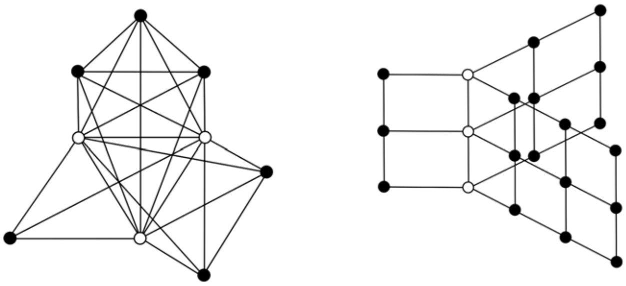
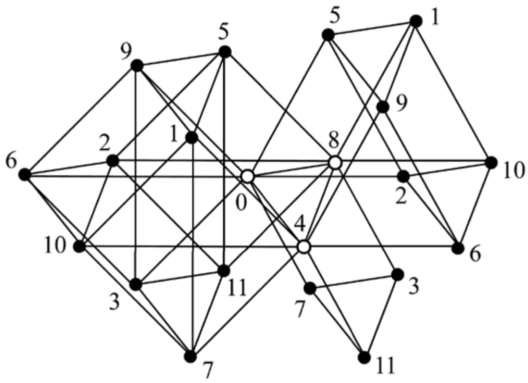


Fig. 1.1. $Amalg(K_3; K_6, K_5, K_4)$ and $Amalg(P_3; P_3 \,\square\, P_4, P_3 \,\square\, P_3, P_3 \,\square\, P_2)$, respectively.



$$\boxed{\begin{matrix} 0 & 9 & 6 & 3 \\ 4 & 1 & 10 & 7 \\ 8 & 5 & 2 & 11 \end{matrix}} \qquad \boxed{\begin{matrix} 0 & 5 & 2 \\ 4 & 9 & 6 \\ 8 & 1 & 10 \end{matrix}} \qquad \boxed{\begin{matrix} 0 & 7 \\ 4 & 11 \\ 8 & 3 \end{matrix}}$$

Fig. 1.2. $L(2, 1)$-labeling of $Amalg(K_3; K_3 \,\square\, K_4, K_3 \,\square\, K_3, K_3 \,\square\, K_2)$ and the corresponding matrix representation.

identifying $G_1, G_2, \ldots, G_p$ at the vertices in the fixed subgraphs isomorphic to G_0 in each $G_1, G_2, \ldots, G_p$, respectively. G_0 is called the *spine* and G_k is called *page k* of G for $k = 1, 2, \ldots, p$.

In [1], general upper bounds for the λ-number of the amalgamation of graphs were established by determining the exact λ-number of the amalgamation of complete graphs along a complete graph. They also provided the exact λ-numbers of the amalgamation of rectangular grids along a certain path, or more specifically, of the Cartesian product of a path and a star with spokes of arbitrary lengths. This focus on the Cartesian product in the context of amalgamations motivated us to investigate the λ-number of the amalgamation of Cartesian products of complete graphs along a complete graph.

Definition 1.2. The *Cartesian product* of two disjoint graphs G and H, denoted by $G \,\square\, H$, is defined as the graph with vertex set given by the Cartesian product of the vertex set of G and the vertex set of H, where two vertices (u, v) and (w, z) are adjacent if and only if either [u, w are adjacent in G and $v = z$] or [v, z are adjacent in H and $u = w$].

Throughout, $p, n_0, n_1, \ldots, n_p$, are all integers greater than or equal to 2, unless otherwise noted. We will study the amalgamation K of Cartesian products of complete graphs along a complete graph, more specifically, $K = Amalg(K_{n_0}; K_{n_0} \,\square\, K_{n_1}, K_{n_0} \,\square\, K_{n_2}, \ldots, K_{n_0} \,\square\, K_{n_p})$ where K_{n_k} is the complete graph on n_k vertices for $k = 0, 1, \ldots, p$. For a fixed $k = 1, 2, \ldots, p$, the vertices in page k, that is, the vertices in $K_{n_0} \,\square\, K_{n_k}$, can be organized in an array format where each vertex will be represented by an ordered triple (i, j, k) with $i = 0, 1, \ldots, n_0 - 1$, and $j = 0, 1, \ldots, n_k - 1$ so that two vertices are adjacent if their triple representations satisfy exactly one of the following conditions:

(i) Both triples agree on the first and last coordinate, respectively.
(ii) Both triples agree on the second and last coordinate, respectively.

The subgraph induced by the vertices in the same row of this array is isomorphic to K_{n_k} and the subgraph induced by the vertices in the same column is isomorphic to K_{n_0}. Furthermore, for a fixed i, the vertices $(i, 0, k)$ for $k = 1, 2, \ldots, p$, represent the same vertex s_i in the spine K_{n_0}. For convenience, $L(2, 1)$-labelings of K will be represented by the n_0-by-n_k matrices, $k = 1, 2, \ldots, p$, where the entry on the ith row, jth column of the kth matrix will be the label of vertex (i, j, k); observe that all the 0th columns of these p matrices must be the same as they contain the labels for the spine.

To illustrate the different amalgamations mentioned in this section, we provide three examples: in Fig. 1.1, an amalgamation of complete graphs along a complete graph (on the left), and an amalgamation of rectangular grids along a path (on the right); in Fig. 1.2, an amalgamation of Cartesian products of complete graphs along a complete graph with an $L(2, 1)$-labeling (on the left) and the corresponding matrix representation (on the right). In each subfigure, the vertices in the spine are in white.

The following result can be used to derive upper bounds for $\lambda(K)$ when $K = Amalg(K_{n_0}; K_{n_0} \square K_{n_1}, K_{n_0} \square K_{n_2}, \ldots, K_{n_0} \square K_{n_p})$.

Result 1.3. ([1]) Let $G_1, G_2, \ldots, G_p$ be $p \geq 2$ graphs each containing a fixed induced subgraph isomorphic to a graph G_0 with q vertices. If G_1 with m vertices is a graph among $G_1, G_2, \ldots, G_p$ with maximum number of vertices and if $G = Amalg(G_0; G_1, G_2, \ldots, G_p)$ has n vertices, then $\lambda(G) \leq n + q - 1$ if $m \leq (n + q)/2$, and $\lambda(G) \leq 2(m - 1)$ otherwise.

Using this result, we obtain $\lambda(K) \leq n_0(n_1 + n_2 + \cdots + n_p - p + 2) - 1$ if $n_1 \leq n_2 + n_3 + \cdots + n_p - p + 2$, and $\lambda(K) \leq 2(n_0 n_1 - 1)$ otherwise, and it can be verified that both upper bounds are smaller than the conjectured bound $\Delta^2(K) = (n_0 + n_1 + \cdots + n_p - p - 1)^2$. In Section 2, we provide an even smaller general upper bound for $\lambda(K)$ and several exact values within particular infinite subfamilies of amalgamations. We will connect this smaller upper bound to the classic minimum makespan scheduling problem. In this problem, we are given a finite collection of jobs and machines, as well as the processing time incurred by scheduling each job in each respective machine. The goal is to find an assignment of jobs to machines that minimizes the *makespan* or total processing time assigned to any machine. The makespan scheduling problem is known to be NP-hard even when restricted to two identical machines [5].

2. The λ-number of the amalgamation of Cartesian products of complete graphs along a complete graph

In Theorem 2.3, we will describe an infinite family of amalgamations of Cartesian products of complete graphs along a complete graph where the λ-number of each amalgamation coincides with the λ-number of one of its pages. This result will be used later in Corollary 2.4 to provide a general upper bound for the λ-number of most amalgamations of Cartesian products of complete graphs along a complete graph with at least 3 vertices. In Corollary 2.6, we present exact λ-numbers for the cases not covered in Corollary 2.4, still considering spines with at least 3 vertices, and extend Theorem 2.3 in Corollary 2.7. Finally, the cases in which the spine has exactly 2 vertices are treated in Theorem 2.9 and Corollary 2.10, where we once more provide some exact values and general upper bounds for the λ-number. As mentioned previously, all the upper bounds found in this section will be smaller than the bounds given at the end of the Introduction.

We begin with the following lemma in which we present some properties of a particular matrix of nonnegative integers that will be the basis for constructing $L(2, 1)$-labelings of certain Cartesian products of complete graphs along a complete graph in Theorem 2.3.

Lemma 2.1. *Let A be the matrix with $n \geq 3$ rows and $m \geq 2$ columns where for $i = 0, 1, \ldots, n-1$ and $j = 0, 1, \ldots, m-1$, the entry in row i, column j is $A_{ij} = (im + j(1-m)) \bmod nm$.*

(i) *There is a one-to-one correspondence between the set of entries of A and the set $\{0, 1, \ldots, nm-1\}$.*

(ii) *For a given $j = 0, 1, \ldots, m-1$, column j of A is the transpose of the j-*rotation *of the n-tuple $(0m + j, 1m + j, \ldots, (n-1)m + j)$, which is defined as the clockwise circular shifting of its entries by j positions. Observe that this j-rotation can be obtained by subtracting jm from each entry modulo nm.*
Note: For the sake of brevity, a column of a matrix and its transpose will be used interchangeably, and thus the term j-rotation will also apply to columns.

(iii) *If x is in column j of A for some $j = 0, 1, \ldots, m-1$, then $(x+1) \bmod nm$ is in column $(j+1) \bmod m$.*

(iv) *For $k = 1, 2, \ldots, n-1$, let $A(k)$ be the matrix obtained from A by replacing each of its columns, except for the first, with its $(k-1)$-rotation. If x and $(x+1) \bmod nm$ are entries in the same row of $A(k)$, then $(m+k-1) \bmod n = 2$, x is in column $m-1$, and $(x+1) \bmod nm$ is in column 0.*

Proof. The definition of A implies that each row is obtained from the previous row by adding m to each entry modulo nm, and each column is obtained from the previous column by adding $1-m$ to each entry modulo nm. Note that this claim is true if we assume that row 0 follows row $n-1$ but we may not assume column 0 follows column $m-1$. Fig. 2.1 shows three examples of A for $n = 4$ where A_m corresponds to $m = 8, 9, 10$.

Let us start by verifying item (i). By the definition of A, it is obvious that each entry of A belongs to $\{0, 1, \ldots, nm-1\}$. On the other hand, if x belongs to $\{0, 1, \ldots, nm-1\}$, then using the Euclidean division we obtain unique nonnegative integers q and r so that $x = qm + r$ with $r < m$. Set $i = (q + r) \bmod n$ and $j = r$. Again using the Euclidean division, there are unique nonnegative integers q' and r' so that $q + r = q'n + r'$ with $r' < n$. Clearly $i = (q + r) \bmod n = r'$ so

$$\begin{aligned} A_{ij} &= (im + j(1-m)) \bmod nm = [r'm + r(1-m)] \bmod nm \\ &= [(r' - r)m + r] \bmod nm = [(q - q'n)m + r] \bmod nm \\ &= [(qm + r) - q'nm] \bmod nm = (qm + r) \bmod nm = x. \end{aligned}$$

The last equality follows since $0 \leq x = qm + r < nm$. Hence, item (i) holds.

To prove item (ii), let us consider an arbitrary $x = tm + j$ for some $t = 0, 1, \ldots, n-1$ and $j = 0, 1, \ldots, m-1$. As in the verification of item (i), by setting $i = (t + j) \bmod n$, we have $A_{ij} = x$, and thus x is in column j of A. In particular, if $t = 0$, then $i = j \bmod n$ and $A_{ij} = j$. Therefore, since each entry in column j of A can be obtained from the previous entry in the same column by adding m modulo nm, column j is the *j-rotation* of the n-tuple $(0m + j, 1m + j, \ldots, (n-1)m + j)$ as desired.

$$A_8 = \begin{array}{|cccccccc|}\hline 0 & 25 & 18 & 11 & 4 & 29 & 22 & 15 \\ 8 & 1 & 26 & 19 & 12 & 5 & 30 & 23 \\ 16 & 9 & 2 & 27 & 20 & 13 & 6 & 31 \\ 24 & 17 & 10 & 3 & 28 & 21 & 14 & 7 \\ \hline \end{array} \quad A_9 = \begin{array}{|ccccccccc|}\hline 0 & 28 & 20 & 12 & 4 & 32 & 24 & 16 & 8 \\ 9 & 1 & 29 & 21 & 13 & 5 & 33 & 25 & 17 \\ 18 & 10 & 2 & 30 & 22 & 14 & 6 & 34 & 26 \\ 27 & 19 & 11 & 3 & 31 & 23 & 15 & 7 & 35 \\ \hline \end{array} \quad A_{10} = \begin{array}{|cccccccccc|}\hline 0 & 31 & 22 & 13 & 4 & 35 & 26 & 17 & 8 & 39 \\ 10 & 1 & 32 & 23 & 14 & 5 & 36 & 27 & 18 & 9 \\ 20 & 11 & 2 & 33 & 24 & 15 & 6 & 37 & 28 & 19 \\ 30 & 21 & 12 & 3 & 34 & 25 & 16 & 7 & 38 & 29 \\ \hline \end{array}$$

Fig. 2.1. Examples of A for $n = 4$ where A_m corresponds to $m = 8, 9, 10$.

$$A(1) = \begin{array}{|cccccc|}\hline 0 & 25 & 20 & 15 & 10 & 5 \\ 6 & 1 & 26 & 21 & 16 & 11 \\ 12 & 7 & 2 & 27 & 22 & 17 \\ 18 & 13 & 8 & 3 & 28 & 23 \\ 24 & 19 & 14 & 9 & 4 & 29 \\ \hline \end{array} \quad A(2) = \begin{array}{|cccccc|}\hline 0 & 19 & 14 & 9 & 4 & 29 \\ 6 & 25 & 20 & 15 & 10 & 5 \\ 12 & 1 & 26 & 21 & 16 & 11 \\ 18 & 7 & 2 & 27 & 22 & 17 \\ 24 & 13 & 8 & 3 & 28 & 23 \\ \hline \end{array} \quad A(3) = \begin{array}{|cccccc|}\hline 0 & 13 & 8 & 3 & 28 & 23 \\ 6 & 19 & 14 & 9 & 4 & 29 \\ 12 & 25 & 20 & 15 & 10 & 5 \\ 18 & 1 & 26 & 21 & 16 & 11 \\ 24 & 7 & 2 & 27 & 22 & 17 \\ \hline \end{array} \quad A(4) = \begin{array}{|cccccc|}\hline 0 & 7 & 2 & 27 & 22 & 17 \\ 6 & 13 & 8 & 3 & 28 & 23 \\ 12 & 19 & 14 & 9 & 4 & 29 \\ 18 & 25 & 20 & 15 & 10 & 5 \\ 24 & 1 & 26 & 21 & 16 & 11 \\ \hline \end{array}$$

Fig. 2.2. Examples of $A(k)$ for $k = 1, 2, 3, 4$ when $n = 5$ and $m = 6$.

$$B(1) = \begin{array}{|cccccc|}\hline 0 & 25 & 20 & 15 & 10 & 5 \\ 6 & 1 & 26 & 21 & 16 & 11 \\ 12 & 7 & 2 & 27 & 22 & 17 \\ 18 & 13 & 8 & 3 & 28 & 23 \\ 24 & 19 & 14 & 9 & 4 & 29 \\ \hline \end{array} \quad B(2) = \begin{array}{|cccccc|}\hline 0 & 19 & 14 & 9 & 4 & 23 \\ 6 & 25 & 20 & 15 & 10 & 29 \\ 12 & 1 & 26 & 21 & 16 & 5 \\ 18 & 7 & 2 & 27 & 22 & 11 \\ 24 & 13 & 8 & 3 & 28 & 17 \\ \hline \end{array} \quad B(3) = \begin{array}{|cccccc|}\hline 0 & 13 & 8 & 3 & 28 & 17 \\ 6 & 19 & 14 & 9 & 4 & 23 \\ 12 & 25 & 20 & 15 & 10 & 29 \\ 18 & 1 & 26 & 21 & 16 & 5 \\ 24 & 7 & 2 & 27 & 22 & 11 \\ \hline \end{array} \quad B(4) = \begin{array}{|cccccc|}\hline 0 & 7 & 2 & 27 & 22 & 11 \\ 6 & 13 & 8 & 3 & 28 & 17 \\ 12 & 19 & 14 & 9 & 4 & 23 \\ 18 & 25 & 20 & 15 & 10 & 29 \\ 24 & 1 & 26 & 21 & 16 & 5 \\ \hline \end{array}$$

Fig. 2.3. Examples of $B(k)$ for $k = 1, 2, 3, 4$ in Theorem 2.3 when $n = 5$ and $m = 6$.

To verify item (iii), consider an arbitrary x in column j of A for some $j = 0, 1, \ldots, m-1$. By item (ii), $x = tm + j$ for some $t = 0, 1, \ldots, n-1$. If $j \neq m-1$, then $x + 1 = tm + (j+1)$ which is in column $j+1$ of A again by item (ii). Similarly, if $j = m-1$ and $t \neq n-1$, then $x + 1 = (t+1)m + 0$ so $x + 1$ is in column $0 = (j+1) \bmod m$. Finally, if $j = m-1$ and $t = n-1$, then $(x+1) \bmod nm = nm \bmod nm = 0$, which is also in column $0 = (j+1) \bmod m$. Therefore, item (iii) is true.

We close the proof by verifying item (iv). For $k = 1, 2, \ldots, n-1$, let $A(k)$ be the matrix obtained from A by replacing each of its columns, except for the first, with its $(k-1)$-rotation. Obviously, $A(1) = A$. Fig. 2.2 shows examples of $A(k)$ for $k = 1, 2, 3, 4$ when $n = 5$ and $m = 6$.

Let k be an arbitrary integer in $\{1, 2, \ldots, n-1\}$ and suppose x and $(x+1) \bmod nm$ are entries in the same row of $A(k)$. By item (iii), x is in column j of $A(k)$, and $(x+1) \bmod nm$ is in column $(j+1) \bmod m$ for some $j = 0, 1, \ldots, m-1$. If $j \neq 0$ and $j \neq m-1$, since each column of A is obtained from the previous column by adding $1-m$ to each entry modulo nm, we must have $(x+1) \bmod nm = [x + (1-m)] \bmod nm$, and therefore nm divides $(x+1) - [x + (1-m)] = m$ which is not possible as $n \geq 3$. If $j = 0$, since the second column of $A(k)$ is obtained from the first by adding $1-m$ to each entry modulo nm and then subtracting $(k-1)m$ from each entry modulo nm, we must have $(x+1) \bmod nm = [x + (1-m) - (k-1)m] \bmod nm$ and therefore nm divides $(x+1) - [x + (1-m) - (k-1)m] = km$ which is not possible as $1 \leq k \leq n-1$. Therefore $j = m-1$. We have that column $j = m-1$ of $A(k)$ is the $(k-1)$-rotation of column $m-1$ of A. So by item (ii), column $j = m-1$ is the $(m+k-2)$-rotation of the tuple $(m-1, 2m-1, 3m-1, \ldots, nm-1)$ and column $(j+1) \bmod m = 0$ is the tuple $(0, m, 2m, \ldots, (n-1)m)$. Since x and $(x+1) \bmod nm$ are in the same row of A, this is only possible if the former $(m+k-2)$-rotation is either a 1-rotation or, if $m = 2$, a 0-rotation. But $m = 2$ is not possible because it would imply $0 = (m+k-2) \bmod n = k \bmod n = k$ contradicting the choice of a positive k. Hence $(m+k-2) \bmod n = 1$, or equivalently, $(m+k-1) \bmod n = 2$ as desired. For the example in Fig. 2.2, the first and last columns of $A(2)$ are the only ones containing consecutive labels in the same row in any of the four matrices because $(m+k-1) \bmod n = (6+2-1) \bmod 5 = 2$. □

The following result will be used in the proof of Theorem 2.3.

Result 2.2. ([6]) If $m, n \geq 2$, then $\lambda(K_n \square K_m) = 4$ when $n = m = 2$; otherwise, $\lambda(K_n \square K_m) = nm - 1$.

Theorem 2.3. *Let $K = Amalg(K_n; K_n \square K_m, K_n \square K_m, \ldots, K_n \square K_m)$ with p pages where ($n > 3, m = 2$, and $p = n-2$) or ($n, m \geq 3$, and $p = n-1$). Then $\lambda(K) = nm - 1$.*

Proof. Suppose ($n > 3, m = 2$, and $p = n-2$) or ($n, m \geq 3, m > 2$, and $p = n-1$). Since $K_n \square K_m$ is a subgraph of K, from Result 2.2 we have $\lambda(K) \geq \lambda(K_n \square K_m) = nm - 1$. In order to obtain the desired equality, it is sufficient to exhibit an $(nm-1)$-labeling of K. We will accomplish this by constructing p different n by m matrices with entries in $\{0, 1, \ldots, nm-1\}$ and using each matrix to label one of the p pages $K_n \square K_m$ of K.

For each $k = 1, 2, \ldots, p$, let $A(k)$ be the matrix as defined in item (iv) of Lemma 2.1. Let k_0 be the only integer in $\{1, 2, \ldots, n\}$ such that $(m + k_0 - 1) \bmod n = 2$. For $k = 1, 2, \ldots, k_0 - 1$, let $B(k) = A(k)$, and for $k = k_0, k_0 + 1, \ldots, p$, let $B(k)$ be the matrix obtained from $A(k)$ by replacing its last column with its 1-rotation. Fig. 2.3 shows examples of $B(k)$ for $k = 1, 2, 3, 4$ when $n = 5$, $m = 6$, and $k_0 = 2$ (refer to Fig. 2.2 for the corresponding $A(k)$ for $k = 1, 2, 3, 4$).

Let us consider a fixed k in $\{1, 2, \ldots, p\}$. We use the first column of matrix $B(k)$ to label the spine of K, that is, assign the entry in row i, column 0 of $B(k)$ to vertex $s_i = (i, 0, k)$ in the spine K_n for $i = 0, 1, \ldots, n-1$. Note that there will be no conflict labeling the spine for different values of k since all $A(k)$, and consequently all $B(k)$, coincide on the first column. Each one of the remaining $m-1$ columns of $B(k)$ is used to label one of the $m-1$ complete subgraphs isomorphic to the spine induced by the vertices (i, j, k) for $i = 0, 1, \ldots, n-1$ and a fixed j in $\{1, 2, \ldots, m-1\}$, where the entry in row i, column j of $B(k)$ is assigned to vertex (i, j, k). We will first show that this labeling is an $L(2, 1)$-labeling of page k of K. In view of item (i) in Lemma 2.1, all the labels in $B(k)$ are different, so two vertices at distance at most two in page k are assigned different labels. It remains to be shown that adjacent vertices in page k are not assigned consecutive labels. By construction, two adjacent vertices in page k must get both labels in the same column of $B(k)$, or both labels in the same row of $B(k)$. Two labels in the same column of $B(k)$ differ by at least $m \geq 2$ by the definition of $B(k)$ because each row of the original matrix A is obtained from the previous row by adding m to each entry modulo nm. To show that two labels x and y in the same row of $B(k)$ will also differ by at least 2, we need to break up the discussion into three cases: $k < k_0$, $k = k_0$ and $k > k_0$. If $k < k_0$, then $B(k) = A(k)$ and $(m + k - 1) \bmod n \neq 2$, so x and y differ by at least 2 as implied by item (iv) in Lemma 2.1. If $k = k_0$, then $(m + k_0 - 1) \bmod n = 2$ and $B(k_0)$ agrees with $A(k_0)$ in all but its last column which is by definition $((n-1)m-1, nm-1, m-1, 2m-1, 3m-1, \ldots, (n-2)m-1)$, i.e., the 1-rotation of the last column of $A(k_0)$ which is exactly a 2-rotation of the tuple $(m-1, 2m-1, 3m-1, \ldots, nm-1)$ (recall the proof of item (iv) of Lemma 2.1). In this case, without loss of generality, x is in the last column of $B(k_0)$ and y is in either the first column, which is $(0, m, 2m, \ldots, (n-1)m)$, or in the next to last column, $(m-2, 2m-2, 3m-2, \ldots, (n-1)m-2, nm-2)$. By inspection, if y is in the first column of $B(k_0)$, then x and y differ by at least $m+1 \geq 2$ while if y is in the next to last column of $B(k_0)$, then x and y differ by at least $2m-1 \geq 2$, both contradicting the choice of labels. Thus we again conclude that x and y differ by at least 2. We will finally assume $k > k_0$. By item (iv) of Lemma 2.1, if neither x nor y is in the first or last column of $B(k)$, then they differ by at least 2. Assume on the other hand that at least one of x or y is in the first or last column of $B(k)$ and suppose for contradiction that x and y differ by at most 1. By item (iii) of Lemma 2.1, we need to discuss three different cases depending on which columns contain these labels and show that none of these cases is possible.

Case 1: The first column of $B(k)$, which is $(0, m, 2m, \ldots, (n-1)m)$, contains x and the second column of $B(k)$ contains y. This second column is either

(I) the k-rotation of $(0m + 1, 1m + 1, 2m + 1, \ldots, (n-2)m + 1, (n-1)m + 1)$ if $m > 2$, or
(II) the $(k+1)$-rotation of $(1, 3, 5, \ldots, 2n-3, 2n-1)$ if $m = 2$.

By inspection, if (I) holds, x and y differ by at most 1 only if $k \bmod n = 0$ which is impossible since $k_0 < k < p = n-1$. On the other hand, if (II) holds, we must have $k_0 = 1$ and the second column of $B(1)$ is $(2n-3, 2n-1, 1, 3, 5, \ldots, 2n-5)$, i.e., the 1-rotation of the last column of $A(1)$. Note that this tuple is well defined since $m = 2$ implies $n > 3$ (recall the initial assumption ($n > 3$, $m = 2$, and $p = n-2$) or ($n, m \geq 3$, and $p = n-1$)). In this case, x and y differ by at most 1 only if $(k+1) \bmod n \leq 1$, which is also impossible since $k_0 < k < p = n-2$.

Case 2: The first column of $B(k)$, which is $(0, m, 2m, \ldots, (n-1)m)$, contains x and the last column of $B(k)$ contains y. By items (ii) and (iv) of Lemma 2.1 and the definition of $B(k)$, its last column is the $(m+k-1)$-rotation of the tuple $(m-1, 2m-1, 3m-1, \ldots, nm-1)$. By inspection, x and y differ by at most 1 if this $(m+k-1)$-rotation is either a 1-rotation or, if $m = 2$, a 0-rotation. But $m = 2$ would imply $0 = (m+k-1) \bmod n = (k+1) \bmod n = k+1$ (the last equality holds by recalling that $m = 2$ implies $p = n-2$, and thus $k+1 \leq p+1 = n-1$), which is not possible as k is positive. Hence $m \neq 2$ and the $(m+k-1)$-rotation must be a 1-rotation so $(m+k-1) \bmod n = 1$. This is not possible either since $1 \leq k_0 < k \leq p = n-1$ implies $(m+k-1) \bmod n > (m+k_0-1) \bmod n = 2$.

Case 3: The last column of $B(k)$ contains x and the next-to-last column of $B(k)$ contains y. By items (ii) and (iv) of Lemma 2.1 and the definition of $B(k)$, its next-to-last column is the $(m+k-3)$-rotation of the tuple $(m-2, 2m-2, 3m-2, \ldots, nm-2)$. By comparing to the last column of $B(k)$ (as described in Case 2), x and y differ by at most 1 if this $(m+k-3)$-rotation will place the entry $m-2$ in the same row as the entry $m-1$ on the last column of $B(k)$ which is the $(m+k-1)$-rotation of the tuple $(m-1, 2m-1, 3m-1, \ldots, nm-1)$. So $(m+k-3) = (m+k-1) \mod n$ or, equivalently, n divides $(m+k-1) - (m+k-3) = 2$, which is not possible as $n \geq 3$.

To conclude that combining all the $L(2, 1)$-labelings of pages $k = 1, 2, \ldots, p$ of K results in an $L(2, 1)$-labeling of the entire K, it is sufficient to show that given u and v vertices at distance two in different pages of K, u and v must be assigned different labels. Such u and v are both adjacent to the same vertex in the spine so their labels are on the same row of $B(k)$ and $B(k')$, respectively, for some distinct integers k and k' in $\{1, 2, \ldots, p\}$. We will assume without loss of generality that $k < k'$. By item (i) of Lemma 2.1, each label in $\{0, 1, \ldots, nm-1\}$ appears exactly once in $B(k)$, exactly once in $B(k')$, and in the same column j in $\{1, 2, \ldots, m-1\}$. Moreover, the column j in $B(k)$ (resp., $B(k')$), except for the first column, is either

(III) the $(j+k-1)$-rotation (resp., $(j+k'-1)$-rotation) of $(0m+j, 1m+j, \ldots, (n-1)m+j)$ if $j \neq m-1$ or $k < k_0$ (resp., $k' < k_0$), or
(IV) the $(j+k)$-rotation (resp., $(j+k')$-rotation) of $(0m+j, 1m+j, \ldots, (n-1)m+j)$, otherwise.

Let us suppose for contradiction that $B(k)$ and $B(k')$ coincide in a particular entry in row i in $\{0, 1, \ldots, n-1\}$ and column j. If $j \neq m-1$ or ($k < k_0$ and $k' < k_0$), then the $(j+k-1)$-rotation and $(j+k'-1)$-rotation in (III) must coincide so n must divide $(j+k'-1) - (j+k-1) = k'-k$ which is impossible since $1 \leq k < k' \leq n-1$. If $j = m-1$, $k \geq k_0$, and $k' \geq k_0$, then the $(j+k)$-rotation and $(j+k')$-rotation in (IV) must coincide so $(j+k'-1) - (j+k-1) = k'-k$ which is impossible as remarked previously. The only case left to be examined is if $j = m-1$ and $k < k_0 \leq k'$. In this case, the $(j+k-1)$-rotation

$$B(1)=\begin{array}{|cc|}\hline 0 & 7\\ 2 & 9\\ 4 & 1\\ 6 & 3\\ 8 & 5\\ \hline\end{array}\quad B(2)=\begin{array}{|cc|}\hline 0 & 5\\ 2 & 7\\ 4 & 9\\ 6 & 1\\ 8 & 3\\ \hline\end{array}\quad B(3)=\begin{array}{|cc|}\hline 0 & 3\\ 2 & 5\\ 4 & 7\\ 6 & 9\\ 8 & 1\\ \hline\end{array}\qquad C(1)=\begin{array}{|cc|}\hline 1 & 8\\ 3 & 0\\ 5 & 2\\ 7 & 4\\ \hline\end{array}\quad C(2)=\begin{array}{|cc|}\hline 1 & 6\\ 3 & 8\\ 5 & 0\\ 7 & 2\\ \hline\end{array}\quad C(3)=\begin{array}{|cc|}\hline 1 & 4\\ 3 & 6\\ 5 & 8\\ 7 & 0\\ \hline\end{array}$$

Fig. 2.4. Examples of $B(k)$ and $C(k)$ for $k = 1, 2, 3$ in Corollary 2.6 when $n = 4$.

and $(j + k')$-rotation in (IV) must coincide, so n must divide $(j + k') - (j + k - 1) = k' - k + 1$ which is again impossible since $1 \le k < k' \le n - 1$. Therefore, u and v must be assigned different labels, and the proof is complete. □

The next two corollaries provide a general upper bound and some exact values for $\lambda(K)$, respectively, for all amalgamations K of Cartesian products of complete graphs along a complete graph when the spine has at least three vertices. We say that $X_1, X_2, \dots, X_q$ is a *q-partition* of a set Y if $X_1, X_2, \dots, X_q$ are pairwise disjoint (possibly empty) sets whose union is equal to Y.

Corollary 2.4. *Let $K = Amalg(K_{n_0}; K_{n_0} \square K_{n_1}, K_{n_0} \square K_{n_2}, \dots, K_{n_0} \square K_{n_p})$ where $n_0 \ge 3$, $p \ge 2$ and $n_i \ge 2$ for $i = 1, 2, \dots, p$ so that $K \ne Amalg(K_{n_0}; K_{n_0} \square K_2, K_{n_0} \square K_2, \dots, K_{n_0} \square K_2)$ with $n_0 - 1$ pages. If $n_1 = n_2 = \cdots = n_p = 2$ and $p < n_0 - 1$, let $q = n_0 - 2$; otherwise let $q = n_0 - 1$. If l is the minimum of the $\max_{i=1,2,\dots,q}\{\Sigma_{k\in Xi}(n_k - 1)\}$ over all q-partitions $X_1, X_2, \dots, X_q$ of $\{1, 2, \dots, p\}$, then $\lambda(K) \le n_0(l + 1) - 1$.*

Proof. First suppose $n_1 = n_2 = \cdots = n_p = 2$ and $p < n_0 - 1$. We have $q = n_0 - 2$, so consider the q-partition $X_1, X_2, \dots, X_q$ of $\{1, 2, \dots, p\}$ where $X_i = \{i\}$ for $i = 1, 2, \dots, p$ and $X_i = \varnothing$ for $i = p + 1, p + 2, \dots, q$. Therefore $1 \le l \le \max_{i=1,2,\dots,q}\{\Sigma_{k\in Xi}(n_k - 1)\} = 1$, i.e., $l = 1$. Setting $n = n_0 > 3$ in Theorem 2.3, we have $K' = Amalg(K_{n_0}; K_{n_0} \square K_2, K_{n_0} \square K_2, \dots, K_{n_0} \square K_2)$ with $n_0 - 1$ pages and $\lambda(K') = 2n_0 - 1$. But K is a subgraph of K' hence, $\lambda(K) \le \lambda(K') = 2n_0 - 1 = n_0(l+1) - 1$.

On the other hand, suppose $p \ge n_0 - 1$ or that there exists $j = 1, 2, \dots, p$ such that $n_j > 2$. By definition, $q = n_0 - 1$. If $p > n_0 - 1$, then for any q-partition $X_1, X_2, \dots, X_q$ of $\{1, 2, \dots, p\}$, there exists $t = 1, 2, \dots, q$ such that X_t contains at least two integers. Hence, $\max_{i=1,2,\dots,q}\{\Sigma_{k\in Xi}(n_k - 1)\} \ge \Sigma_{k\in Xt}(n_k - 1) \ge 2$ so $l \ge 2$. If $p = n_0 - 1$, then there exists $j = 1, 2, \dots, p$ such that $n_j > 2$ since we are assuming $K \ne Amalg(K_{n_0}; K_{n_0} \square K_2, K_{n_0} \square K_2, \dots, K_{n_0} \square K_2)$ with $n_0 - 1$ pages. In this case, for any q-partition $X_1, X_2, \dots, X_q$ of $\{1, 2, \dots, p\}$, we must have $\max_{i=1,2,\dots,q}\{\Sigma_{k\in Xi}(n_k - 1)\} \ge n_j - 1 \ge 2$ so we also have $l \ge 2$. Setting $n = n_0 \ge 3$ and $m = l + 1 \ge 3$ in Theorem 2.3, we have that if $K'' = Amalg(K_{n_0}; K_{n_0} \square K_{l+1}, K_{n_0} \square K_{l+1}, \dots, K_{n_0} \square K_{l+1})$ with $n_0 - 1$ pages, then $\lambda(K'') = n_0(l + 1) - 1$. But K is a subgraph of K'' hence, $\lambda(K) \le \lambda(K'') = n_0(l + 1) - 1$. □

It is not difficult to verify that the upper bound in Corollary 2.4 is smaller than the bounds derived from [1] mentioned at the end of the Introduction. The problem of determining l in Corollary 2.4 is computationally complex as it is equivalent to the NP-hard minimum makespan scheduling problem for identical machines where there are p jobs, q identical machines, and the processing time incurred by scheduling job i in any machine is $n_i - 1$, for $i = 1, 2, \dots, p$.

We use the following result in determining the λ-number for one more case with a spine of at least three vertices where the upper bound in Corollary 2.4 does not apply.

Result 2.5. ([9]) If a graph contains three vertices with maximum degree $\Delta \ge 2$ and one of them is adjacent to the other two vertices, then its λ-number is at least $\Delta + 2$.

Corollary 2.6. *Let $n \ge 3$ and $K = Amalg(K_n; K_n \square K_2, K_n \square K_2, \dots, K_n \square K_2)$ with $n - 1$ pages. Then $\lambda(K) = 2n$.*

Proof. Since $\Delta(K) = 2n - 2$ is achieved by the $n \ge 3$ vertices on the spine, $\lambda(K) \ge \Delta(K) + 2 = 2n$ by Result 2.5. To show that $\lambda(K) = \Delta(K) + 2$, it is sufficient to exhibit a $2n$-labeling of K. Suppose $K' = Amalg(K_{n+1}; K_{n+1} \square K_2, K_{n+1} \square K_2, \dots, K_{n+1} \square K_2)$ with $n - 1$ pages. Since $n + 1 > 3$ and the number of pages is exactly $(n + 1) - 2$, Theorem 2.3 implies $\lambda(K') \le 2(n + 1) - 1 = 2n + 1$. Consider the $(2n + 1)$-labeling of K' given by $B(k)$ for $k = 1, 2, \dots, n - 1$, as defined in the proof of Theorem 2.3. Let $C(k)$ be the matrix obtained from $B(k)$ by deleting its first row and subtracting 1 from each of its entries, for $k = 1, 2, \dots, n - 1$. Fig. 2.4 contains examples of $B(k)$ and $C(k)$ for $k = 1, 2, 3$ when $n = 4$.

Note that the label 0 appears only once in each $B(k)$ and is in its first row, therefore $C(k)$ will only contain entries in $\{0, 1, \dots, 2n\}$. For $i = 1, 2, \dots, n - 1$, $j = 1, 2$, and $k = 1, 2, \dots, n - 1$, assign the entry in row i, column j of $C(k)$ to vertex (i, j, k) of K. This labeling does not violate the distance conditions by construction so we can conclude that it is a $2n$-labeling of K as desired. □

Unfortunately, the upper bound given in Corollary 2.4 may still be significantly larger than the actual $\lambda(K)$. For example, if $K = Amalg(K_3; K_3 \square K_4, K_3 \square K_4, K_3 \square K_2)$, then in Corollary 2.4 we have $n = 3$, $q = 2$, $l = 4$, and $\lambda(K) \le 14$. However, $\lambda(K) = 11$ by Corollary 2.7.

Corollary 2.7. *Let $n \ge 3$ and $m \ge 4$. If $K = Amalg(K_n; K_n \square K_m, K_n \square K_m, \dots, K_n \square K_m, K_n \square K_{m-2})$ has n pages with $n - 1$ of them isomorphic to $K_n \square K_m$, then $\lambda(K) = nm - 1$.*

$$B(5)=\begin{array}{|cccc|}\hline 0 & 26 & 21 & 16\\ 6 & 2 & 27 & 22\\ 12 & 8 & 3 & 28\\ 18 & 14 & 9 & 4\\ 24 & 20 & 15 & 10\\ \hline\end{array}$$

Fig. 2.5. $B(5)$ defined in Corollary 2.7 when $n=5$ and $m=6$.

$$M(k)=\begin{array}{|ccccccc|}\hline 0 & k+1 & p+k+1 & 2p+k+1 & \dots & (m-3)p+k+1 & (m-2)p+k+1\\ (m-1)p+2 & p+k & 2p+k & 3p+k & \dots & (m-2)p+k & k\\ \hline\end{array}$$

Fig. 2.6. $M(k)$ used in Theorem 2.9 to label page $k=1,2,\dots,p$ of $K=Amalg(K_2;K_2\,\square\,K_m,K_2\,\square\,K_m,\dots,K_2\,\square\,K_m)$ when $m>2$ and $p>2$.

$$N(1)=\begin{array}{|cc|}\hline 0 & 4\\ 5 & 2\\ \hline\end{array}\quad N(2)=\begin{array}{|cc|}\hline 0 & 3\\ 5 & 1\\ \hline\end{array}\qquad P(1)=\begin{array}{|ccc|}\hline 0 & 2 & 5\\ 7 & 4 & 1\\ \hline\end{array}\quad P(2)=\begin{array}{|ccc|}\hline 0 & 3 & 6\\ 7 & 5 & 2\\ \hline\end{array}$$

Fig. 2.7. $N(k)$ (resp., $P(k)$) used in Theorem 2.9 to label page $k=1,2$ of $K=Amalg(K_2;K_2\,\square\,K_m,K_2\,\square\,K_m)$ when $m=2$ (resp., $m=3$).

$$Q(1)=\begin{array}{|cccccc|}\hline 0 & 2 & 4 & 6 & \dots & 2m-2\\ 2m & 2m-3 & 1 & 3 & \dots & 2m-5\\ \hline\end{array}\qquad Q(2)=\begin{array}{|cccccc|}\hline 0 & 3 & 5 & 7 & \dots & 2m-1\\ 2m & 2m-2 & 2 & 4 & \dots & 2m-4\\ \hline\end{array}$$

Fig. 2.8. $Q(k)$ used in Theorem 2.9 to label page $k=1,2$ of $K=Amalg(K_2;K_2\,\square\,K_m,K_2\,\square\,K_m)$ when $m>3$.

Proof. Let $K'=Amalg(K_n;K_n\,\square\,K_m,K_n\,\square\,K_m,\dots,K_n\,\square\,K_m)$ with $n-1$ pages. Obviously, K' is a subgraph of K so $\lambda(K)\geq\lambda(K')=nm-1$ by Theorem 2.3. Recall the $(nm-1)$-labeling of K' given by the matrices $B(k)$ for $k=1,2,\dots,n-1$ as defined in the proof of Theorem 2.3. Define $B(n)$ to be the matrix obtained from $B(n-1)$ by 1-rotating each of its columns, except for column 0, and then deleting columns 1 and $m-1$. Extend the labeling of K' to a labeling of K, by assigning the entry in row i, column j of $B(n)$ to vertex (i,j,n) in the last page $K_n\,\square\,K_{m-2}$ of K. Techniques similar to the ones in the proof of Theorem 2.3 can be used to show that this extended labeling is an $(nm-1)$-labeling of K. We leave the details to the reader for the sake of brevity. Fig. 2.5 shows $B(5)$ when $n=5$ and $m=6$ (refer to Fig. 2.3 for the corresponding $B(k)$ for $k=1,2,3,4$). □

Up to this point, the focus was on the amalgamation of Cartesian products of complete graphs along a complete graph with at least 3 vertices. We will now focus on the remaining instances where the spine contains exactly 2 vertices. The following result will be used in the proof of Theorem 2.9.

Result 2.8. ([1]) Let $G=Amalg(P_2;P_2\,\square\,P_{n_1},P_2\,\square\,P_{n_2},\dots,P_2\,\square\,P_{n_p})$ with $p\geq 3$, $n_k\geq 2$ for $k=1,2,\dots,p$, where P_i is the path on i vertices. Then $\lambda(G)=6$ if $p=3$; otherwise, $\lambda(G)=p+2$.

The techniques used to show that the labelings presented in the proof of the next result are $L(2,1)$-labelings are similar to the ones used previously so the details will be left to the reader for the sake of brevity.

Theorem 2.9. *Let $K=Amalg(K_2;K_2\,\square\,K_m,K_2\,\square\,K_m,\dots,K_2\,\square\,K_m)$ with $p\geq 2$ pages and $m\geq 2$. Then $\lambda(K)=\Delta(K)+2$ if ($m=2$ and $p=2,3$) or ($m=3$ and $p=2$); otherwise, $\lambda(K)=\Delta(K)+1$.*

Proof. First note that $\Delta(K)=(m-1)p+1$. Let us examine the case $p>2$. If $m=2$, then $\Delta(K)=p+1$ and since $K_2=P_2$, Result 2.8 implies $\lambda(K)=6=\Delta(K)+2$ if $p=3$, and $\lambda(K)=p+2=\Delta(K)+1$ if $p>3$. If $m>2$, then assign the entry in row i column j of $M(k)$ of Fig. 2.6 to vertex (i,j,k) in page k of K for $k=1,2,\dots,p$. By inspection, this is an $((m-1)p+2)$-labeling of K, so $\lambda(K)\leq(m-1)p+2=\Delta(K)+1$. Obviously, $\lambda(K)\geq\Delta(K)+1$, therefore $\lambda(K)=\Delta(K)+1$.

Now assume $p=2$, that is $K=Amalg(K_2;K_2\,\square\,K_m,K_2\,\square\,K_m)$ so $\Delta(K)=2m-1$. If $m=2$ (resp., $m=3$), assign the entry in row i, column j of $N(k)$ (resp., $P(k)$) of Fig. 2.7 to vertex (i,j,k) in page k of K, for $k=1,2$. By inspection, this is a 5-labeling (resp., 7-labeling) of K so $\lambda(K)\leq 5=\Delta(K)+2$ if $m=2$ (resp., $\lambda(K)\leq 7=\Delta(K)+2$ if $m=3$). An exhaustive case discussion shows that $\lambda(K)=\Delta(K)+2$ when $m=2,3$. If $m>3$, then assign the entry in row i, column j of $Q(k)$ of Fig. 2.8 to vertex (i,j,k) in page k of K, for $k=1,2$. By inspection, this is a $(\Delta(K)+1)$-labeling of K so $\lambda(K)\leq\Delta(K)+1$. The equality holds as it is obvious that $\lambda(K)\geq\Delta(K)+1$. □

Corollary 2.10. *Let $K=Amalg(K_2;K_2\,\square\,K_{n_1},K_2\,\square\,K_{n_2},\dots,K_2\,\square\,K_{n_p})$ where $p\geq 2$ and $n_1\geq n_i\geq 2$ for $i=2,3,\dots,p$. Then $\lambda(K)\leq(n_1-1)p+3$ if ($n_1=2$ and $p=2,3$) or ($n_1=3$ and $p=2$); otherwise, $\lambda(K)\leq(n_1-1)p+2$.*

Proof. The desired result follows from Theorem 2.9 by observing that K is a subgraph of $K'=Amalg(K_2;K_2\,\square\,K_{n_1},K_2\,\square\,K_{n_1},\dots,K_2\,\square\,K_{n_1})$ with p pages and $\Delta(K')=(n_1-1)p+1$. □

3. Concluding remarks and directions for future research

We provided exact λ-numbers for two infinite classes of amalgamations of Cartesian products of complete graphs along a complete graph when all the pages are isomorphic: one class in which the spine has exactly two vertices, and the other in which the number of pages is at most the number of vertices in the spine minus one or two. We used these exact λ-numbers to provide upper bounds for the λ-numbers of more general graphs. It would be interesting to determine under what conditions these bounds are tight. Moreover, it may be fruitful to search for other families of amalgamations of graphs where the λ-number coincides with the λ-number of one of its pages as was true here in Theorem 2.3 and Corollary 2.7.

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